# NOTES FOR REPRESENTATION STABILITY OF FINITE ORTHOGONAL GROUPS

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ABSTRACT. abstract comes here

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## 1. INTRODUCTION

The notion of *representation stability* for a sequence of representations $V_n$ of groups $G_n$ satisfying certain consistency conditions was introduced in [CF13] to describe the frequent observation that various representation-theoretic properties stabilized at large $n$. For example, a sequence of ascending topological spaces or groups induces morphisms at the level of (co)homology; in certain cases, these morphisms are eventually isomoprhisms. Now, the study of representation stability is a rich area that lends itself to multiple perspectives, including those from topology, representation theory, category theory, and algebraic geometry.

A framework using the functor category of **FI**-modules was developed in [CEF15] to answer questions about representation stability, particularly as it pertains to the symmetric group, as the endomorphisms in the category **FI** are precisely the symmetric groups (the category **FI** is the category whose objects are finite sets and whose morphisms are injections). The **FI**-modules satisfy a certain Noetherian property, which is critical to the utility of the framework. In particular, it is used to prove an asymptotic structure theorem, which elucidates the stabilization properties observed.

In [PS17], analogues (**VIC**-modules, **SI**-modules) to the functor category of **FI**-modules were constructed by replacing the symmetric groups with the general linear groups and the symplectic groups (over finite rings). Similar Noetherian properties and asymptotic structure theorems were proven, as well as broad homological stability theorems. Some of these results are strengthened in [MW20].

The paper [PS17] did not cover the case of the orthogonal groups; it should be a straightforward generalization of the symplectic case to prove the same results for the orthogonal group. Therefore, for a PRIMES project, we would prove these results for the orthogonal groups and then see if analogues of the techniques in [MW20] also work.

**Remark 1.1.** the orthogonal category has not been studied much in detail; in fact it does not seem to appear anywhere in the literature.

## 2. Reviewing the General Setting

We use the terminology as in [PS17].

**Definition 2.1.** Let $\mathcal{C}$ be a category and $\mathbb{K}$ a ring. Then, a **$\mathcal{C}$-module** over $\mathbb{K}$ is a functor $M : C \to \mathbf{Mod}_{\mathbb{K}}$. If the ring is clear we shall just use the term "$\mathcal{C}$-module". A **$\mathcal{C}$-module homomorphism** $\eta : M \to N$ between two $\mathcal{C}$-modules $M$ and $N$ is a natural transformation of functors.

A $\mathcal{C}$-module homomorphism is injective if each component is injective (resp. surjective). Hence, we can define submodules and quotients; $N$ is a submodule of $M$ is there is an injective $\mathcal{C}$-module homomorphism $N \to M$ and is quotient of $M$ is there is a surjective $\mathcal{C}$-module homorphism $M \to N$. Hence, the category $\mathbf{Mod}_{\mathcal{C}}$ of $\mathcal{C}$-modules is an abelian category.

One of the key ingredients in [CEF15] is a notion of Noetherianity property. Recall that a module of a ring $R$ is Noetherian if every submodule is finitely generated. To generalize that notion to a $\mathcal{C}$-module, we have the following notions:

**Definition 2.2.** A $\mathcal{C}$-module $M$ is **finitely generated** if there exist objects $C_1, C_2, \ldots C_n \in \mathcal{C}$ and elements $x_i \in M(C_i)$ for each $i$ such that if $N$ is a submodule of $M$ such that $N(C_i)$ contains $x_i$, then $N = M$. Let $\{x_i\}$ be called the generating set of $M$.

There is an alternate formulation of this definition. Namley, if $X \in \mathcal{C}$, let $P_{\mathcal{C},X}$ denote the representable $\mathcal{C}$-module generated at $X$ i.e. the functor given by $P_{\mathcal{C},X}(Y) = \mathbb{K}[\mathrm{Hom}_{\mathcal{C}}(X, Y)]$ for all $Y \in \mathcal{C}$. By the Yoneda lemma, a $\mathcal{C}$-module homomorphism $\eta : P_{\mathcal{C},X} \to M$ is determined by a choice of element $x \in M(X)$ and letting $\eta_X(1_X) = x$. Then, a $\mathcal{C}$-module is finitely generated if and only if it is a quotient of a direct sum of modules of the form $P_{\mathcal{C},X}$. Indeed, if $M$ is a finitely generated $\mathcal{C}$-module, then $M$ is a quotient of the direct sum of the representable functors attached to the generating set. Similarly, if $M$ is a quotient of this form, then the elements corresponding to each representable functor will be a generating set of $M$.

We can now define Noetherianity:

**Definition 2.3.** A $\mathcal{C}$-module is **Noetherian** (or **locally Noetherian**) if every submodule is finitely generated, in the sense above. The category of $\mathcal{C}$-modules is **locally Noetherian** if for all Noetherian rings $\mathbb{K}$, all $\mathcal{C}$-modules are Noetherian.

**Lemma 2.4.** *Let $\mathcal{C}$ be a category. The category of $\mathcal{C}$-modules is locally Noetherian if and only if for all $x \in \mathcal{C}$, every submodule of $P_{C,x}$ is finitely generated.*

**Lemma 2.5.** *If the category of $\mathcal{C}$-modules is locally Noetherian, and $f : \mathcal{C} \to \mathcal{D}$ is a finite and essentially surjective functor, then the category of $\mathcal{D}$-modules is locally Noetherian.*

## 3. Reviewing the Symplectic Case

Let $R$ be a finite commutative ring. Recall the category $\mathbf{SI}(R)$:

**Definition 3.1.** A **symplectic module** (over $R$) is a finite-rank free $R$-module $V$ with a non-degenerate alternating form $\omega$ (i.e. $\omega(v,v) = 0$ for all $v \in V$ and the induced map into the dual of $V$ is an isomorphism). Let us use the notation $(V, \omega)$ to denote a symplectic module. Let $\mathbf{SI}(R)$ denote the category whose objects are symplectic modules over $R$ and whose morphisms are $R$-linear maps that preserve the form; these are necesarily injective.

Up to isomorphism, there is only one symplectic module for each even rank and none for odd ranks (c.f. [MH73] for local rings, from which it follows for finite commutative rings):

**Proposition 3.2.** *If $(V, \omega)$ is a symplectic module over a finite commutative ring $R$, then $V$ has even rank and we can find a basis $\{v_{-1}, v_1, v_{-2}, v_2, \dots, v_{-n}, v_n\}$ of $V$ such that $\omega(v_{-i}, v_i) = -\omega(v_i, v_{-i}) = 1$ and $\omega(v_j, v_k) = 0$, where $i, j, k \in \{-n, -(n-1), \dots, n-1, n\}$ and $i > 0, j \neq k$.* □

Therefore, the category $\mathbf{SI}$ is equivalent to the (strict) monoidal category generated by the object $(R^2, \omega_{\text{std}})$ with monoidal operation given by direct sum and unit given by the zero module with zero form. By extending morphisms to the identity on the complement of the form, the map $(V, \omega) \mapsto Sp(V)$ extends to a functor $\mathbf{SI} \to \mathbf{Grp}$. The symplectic groups attached to symplectic modules of a fixed rank are all isomorphic.

In this section, we will review the proof of the following theorem:

**Theorem 3.3.** *The category of* $\mathbf{SI}$*-modules is locally Noetherian.*

**Definition 3.4.** Let $R$ be a commutative local ring. An $R$-linear map $f : R^m \to R^n$ is **column-adapted** if there is a $n$-element subset $S_c(f) = \{s_1 < s_2 < \cdots < s_n\} \subseteq [m]$ such that, if we write $f$ as a $n \times m$ matrix $M$ with respect to the standard basis, then

- The $s_i$th column of $M$ has 1 on the $i$th position and 0 elsewhere;
- The entries $(i, j)$ where $j < s_i$ are all non-invertible.

For example, the map $f : R^5 \to R^3$ defined by the matrix

$$\begin{pmatrix} * & 1 & 0 & \bullet & 0 \\ * & 0 & 1 & \bullet & 0 \\ * & 0 & 0 & * & 1 \end{pmatrix}$$

is column adapted, if the entries labeled with $*$ are non-invertible.

In the general case where $R$ is a finite commutative ring, by Proposition 4.3 there exists an isomorphism

$$R \cong R_1 \times \cdots \times R_q$$

where the $R_i$s are finite commutative local rings. In this case, we say a map $f : R^m \to R^n$ is **column-adapted** if the induced maps $R_i^m \to R_i^n$ are all column-adapted. Also, we say $f$ is **row-adapted** if its transpose is column-adapted; define $S_r(f) = S_c(f^T)$.

**Lemma 3.5.** *The composition of two column-adapted maps is column-adapted. Similarly, the composition of two row-adapted maps is row-adapted.*

The next lemma reveals the importance of column-adapted maps.

**Lemma 3.6.** *Let $R$ be a finite commutative ring, and let $f : R^{n'} \to R^n$ be a surjection. Then we can uniquely factor $f = f_2 f_1$, where $f_1 : R^{n'} \to R^n$ is column-adapted and $f_2 : R^n \to R^n$ is an isomorphism.*

*Proof.* It suffices to deal with local rings.

*Existence:* We know $R^n$ is free, so it is projective, so there exists $g$ such that $fg = 1_{R^n}$. By Cauchy-Binet formula, we have

$$1 = \det fg = \sum_I \det f_I \det g_I$$

where $I$ ranges over all $n$-element subsets of $[n']$, and $f_I$ (resp. $g_I$) denotes $f$ restricted to the columns (resp. rows) in $I$. Because the (unique) maximal ideal of a local ring consists of its non-units, there exists $I$ such that $\det f_I$ is invertible. We take a minimal such $I = \{s_1, \ldots, s_n\}$ (in lexicographic order); since $\det f_I$ is invertible, $f_I$ is invertible as a linear map $R^n \to R^n$, so there exists a $h \in \mathrm{GL}(R^n)$ such that $(hf)_I = hf_I = 1_{R^n}$. If there exists some invertible element in entry $(i, j)$ in the matrix of $hf$ such that $j < s_i$, then the columns $(s_1, \ldots, s_{i-1}, j, s_{i+1}, \ldots, s_n)$ form a basis of $R^n$, contradicting the minimality of $I$. Therefore, $hf$ is column-adapted, and we could take $f_2 = h^{-1}$, $f_1 = hf$ such that $f = f_2 f_1$.

*Uniqueness:* Suppose for contradiction that $f_1 = gf_2$ where $g : R^n \to R^n$ is invertible and $f_1, f_2 : R^{n'} \to R^n$ are column-adapted. Since $g$ is invertible, $\det g$ is invertible, so its last column contains at least one invertible element (recall that the nonunits form an ideal). The definition of column-adapted maps then implies that $\max S_c(f_2) > \max S_c(f_1)$. But we also have $f_2 = g^{-1} f_1$, so $\max S_c(f_1) > S_c(f_2)$, contradiction. □

**Definition 3.7.** Define the category $\mathbf{OSI}'(R)$ by

- objects: pairs $(R^{2n}, \omega)$ where $\omega$ is a symplectic form on $R^{2n}$;
- morphisms: symplectic row-adapted linear maps $R^{2n} \to R^{2n'}$.

Define the category $\mathbf{OSI}(R)$ as the full subcategory of $\mathbf{OSI}'(R)$ spanned by the objects $(R^{2n}, \omega_{\mathrm{std}})$ where $\omega_{\mathrm{std}}$ is the standard symplectic form (described in Proposition 3.2).

| Category | Objects | Morphisms |
|---|---|---|
| $\mathbf{SI}(R)$ | $(V, \omega)$<br>$V$ a finite-rank free $R$-module<br>$\omega$ a non-degenerate alternating form on $V$ | symplectic $R$-linear maps |
| $\mathbf{OSI}'(R)$ | $(R^{2n}, \omega)$<br>$\omega$ a non-degenerate alternating form on $R^{2n}$ | symplectic row-adapted<br>$R$-linear maps |
| $\mathbf{OSI}(R)$ | $(R^{2n}, \omega_{\mathrm{std}})$ | symplectic row-adapted<br>$R$-linear maps |

Applying Lemma 3.6 to the category $\mathbf{SI}(R)$, we obtain the following lemma.

**Lemma 3.8.** *Let $f \in \mathrm{Hom}_{\mathbf{SI}(R)}((R^{2n}, \omega), (R^{2n'}, \omega'))$, then we can uniquely write $f = f_1 f_2$ such that $f_2 : (R^{2n}, \omega) \to (R^{2n}, \gamma)$ is an isomorphism for some symplectic form $\gamma$ on $R^{2n}$, and $f_1 \in \mathrm{Hom}_{\mathbf{OSI}'(R)}((R^{2n}, \gamma) \to (R^{2n'}, \omega))$.*

*Proof.* Transposing, applying Lemma 3.6, then transposing back, we can uniquely write $f = f_1 f_2$, where $f_1 : R^{2n} \to R^{2n'}$ is row-adapted andd $f_2 : R^{2n} \to R^{2n}$ is an isomorphism. We can then uniquely choose $\gamma$ a symplectic form on $R^{2n}$ so that $f_1$ and $f_2$ are symplectic. □

Our goal is to prove that the category of $\mathbf{OSI}(R)$-modules is locally Noetherian, from which we could prove the desired conclusion that the category of $\mathbf{SI}(R)$-modules is locally

Noetherian. To do this, we need the existence of a well partial ordering $\preceq$ on the set

$$\mathcal{P}_R(d,\omega) = \bigsqcup_{n\geq 0} \mathrm{Hom}_{\mathbf{OSI}'(R)}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$$

as described in the following lemma.

**Lemma 3.9.** *Fix $R, d, \omega$. There exists a well partial ordering $\preceq$ on $\mathcal{P}_R(d,\omega)$ that can be extended to a total ordering $\leq$ such that for $f, g \in \mathcal{P}_R(d,\omega)$, mapping to $R^{2n}, R^{2n'}$ respectively, satisfying $f \preceq g$, there exists some $\phi \in \mathrm{Hom}_{\mathbf{OSI}(R)}((R^{2n},\omega_{\mathrm{std}}),(R^{2n'},\omega_{\mathrm{std}}))$ such that:*

- *$g = \phi f$;*
- *For any $f_1 \in \mathrm{Hom}_{\mathbf{OSI}'(R)}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$ with $f_1 < f$, $\phi f_1 < g$.*

The construction of the partial order relies on the next lemma:

**Lemma 3.10.** *Fix $R, d, \omega, f, g, n, n'$ as described in Lemma 3.9, and let the rows of $g$ be $r_1, \ldots, r_{2n'}$. Suppose that $f$ can be obtained from $g$ by deleting certain rows $r_j$, $j \in J = \{2i-1, 2i\}$ where $i$ ranges over some subset of $[n']$ such that $J \cap S_r(g) = \varnothing$. Then there exists $\phi \in \mathrm{Hom}_{\mathbf{OSI}(R)}(R^{2n}, R^{2n'})$ such that:*

- *For any $h \in \mathrm{Hom}_{\mathbf{OSI}'(R)}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$ with $S_r(h) = S_r(f)$, $\phi h$ can be obtained from $h$ by inserting the row $r_j$ in position $j$ for all $j \in J$. In particular, $g = \phi f$.*
- *For any $h \in \mathrm{Hom}_{\mathbf{OSI}'(R)}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$ with $S_r(h) < S_r(f)$ in lex order, then $S_r(\phi h) < S_r(g)$ in lex order.*

*Proof.* The desired map $\phi$ can be defined by the following $2n' \times 2n$ matrix: take a $2(n'-n) \times 2n$ matrix where the $k$th row $\widehat{r_k}$ coincides with $r_k$ only at positions in $S_r(f)$ and zeros elsewhere, and shuffle its rows with the rows of a $2n \times 2n$ identity matrix, such that the former rows occupy row indices in $J$. It is straightforward to check that the two points actually hold; the nontrivial part is to prove that $\phi$ preserves $\omega_{\mathrm{std}}$, which can be proven by carefully analyzing the columns of $f, g, \phi$ and using the assumptions that $f, g$ are symplectic maps. □

We now prove Lemma 3.9 in the case where $R$ is a local ring. We will use the following lemma without proof.

**Lemma 3.11.** *Let $\Sigma$ be a set, then $\Sigma^\star$ denotes the set of words in $\Sigma$. The set $\Sigma^\star$ can be made into a poset by declaring that $s_1 s_2 \ldots s_n \leq s'_1 s'_2 \ldots s'_m$ if there is an increasing function $f : [n] \to [m]$ such that $s_i = s'_{f(i)}$ for all $i$. Then $\Sigma^\star$ is well-ordered poset.*

*Proof of Lemma 3.9, local case.* Let $f, g \in \mathcal{P}_R(d,\omega)$ mapping to $2n, 2n'$ respectively, we declare $f \preceq g$ if it can be obtained from $g$ by deleting some set of rows $J = \{2i-1, 2i\}$ where $i$ ranges in a subset of $[n']$ such that $J \cap S_r(g) = \varnothing$. This is clearly a partial order.

We now prove that $(\mathcal{P}_R(d,\omega), \preceq)$ is isomorphic to a subposet of $\Sigma^*$, where we take $\Sigma = (R^d \sqcup \{\bullet\}) \times (R^d \sqcup \{\bullet\})$, $\bullet$ being a formal symbol. This would imply that $\preceq$ is a well partial ordering. For $f \in \mathrm{Hom}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$, let $r_i$ represent the $i$th row of $f$ it $i \notin S_r(f)$, else let $r_i = \bullet$. Thus, each pair $\theta_i = (r_{2i-1}, r_{2i}) \in \Sigma$, and we map $f$ to the word $\theta_1\theta_2\ldots\theta_n \in \Sigma^*$. Clearly this map is an order-preserving injection, so we conclude that $\preceq$ is a well partial ordering.

Next, we extend $\preceq$ to a total ordering $\leq$. Fix an arbitrary total order on $R^{2d}$, for $f \neq g$ the order is defined by

- If $n < n'$ then $f < g$;
- Otherwise, if $S_r(f) < S_r(g)$ in lex order, then $f < g$;
- Otherwise, compare the sequences of rows of $f$ and $g$ in lexicographic order and the total order on $R^{2d}$.

This clearly extends $\preceq$, and the claimed properties follow by taking $\phi$ as described by Lemma 3.10. □

This can be extended to the case where $R$ is a finite commutative ring.

*Proof of Lemma 3.9, general case.* Fix $R \cong R_1 \times \cdots \times R_q$, then

$$\mathrm{Hom}((R^{2n}, \omega), (R^{2n'}, \omega_{\mathrm{std}})) = \prod_{i=1}^{q} \mathrm{Hom}(R_i^{2n}, \omega|_{R_i}), (R_i^{2n'}, \omega_{\mathrm{std}})|_{R_i}.$$

Thus, the set $\mathcal{P}_R(d, \omega)$ can be identified with

$$\mathcal{P}_{R_1}(d, \omega|_{R_1}) \times \cdots \times \mathcal{P}_{R_q}(d, \omega|_{R_q}).$$

It is then easy to define a partial order $\preceq$ and extend it into a total order $\leq$ using lexicographic order. □

Using this well ordering, we can deduce that:

**Theorem 3.12.** *Let $R$ be a finite commutative ring. For $d \geq 0$ and $\omega$ a symplectic form on $R^{2d}$, any $\mathbf{OSI}(R)$-submodule of the $\mathbf{OSI}(R)$-module*

$$Q_{d,\omega} = \mathbb{K}[\mathrm{Hom}_{\mathbf{OSI}'(R)}((R^{2d}, \omega), -)]$$

*is finitely generated. As a corollary, the category of $\mathbf{OSI}(R)$-modules is locally Noetherian.*

*Proof.* In view of Lemma 2.4, it suffices to prove that any submodule of $Q_{d,\omega}$ is finitely generated.

Fix $d, \omega, R, \mathbb{K}$, so we abbreviate $Q_{d,\omega}$ as $Q$. For an element $f \in \mathrm{Hom}((R^{2d}, \omega), R^{2n})$, let $e_f$ denote the basis vector in $Q(R^{2n})$ corresponding to $f$. For an element $x \in Q(R^{2n})$, define its **initial term** $\mathrm{init}(x)$ as follows: if $f$ is $\leq$-maximal such that $e_f$ has coefficient $\alpha_f \neq 0$ in $x$, $\mathrm{init}(x) = \alpha_f e_f$. Let $M$ be a submodule of $Q$, we also define $\mathrm{init}(M)$ to be a function taking $R^{2n}$ to the $\mathbb{K}$-module $\mathbb{K}[\mathrm{init}(x) \mid x \in M(R^{2n})]$.

We claim that if $N$ is a submodule of $M$ and $N \neq M$, then $\mathrm{init}(N) \neq \mathrm{init}(M)$. Suppose for contradiction that $\mathrm{init}(N) = \mathrm{init}(M)$. Pick $y \in M(R^{2n}) \backslash N(R^{2n})$ such that $\mathrm{init}(y) = \alpha_t e_t$ is $\leq$-minimal. Since $\mathrm{init}(M) = \mathrm{init}(N)$, there exists $z \in N(R^{2n})$ such that $\mathrm{init}(z) = \mathrm{init}(y)$, but then $z - y \notin N(R^{2n})$ and $\mathrm{init}(z - y)$ is smaller than $e_t$, contradiction. This proves the claim.

Suppose now that there exists a increasing sequence of submodules of $Q$

$$M_0 \subsetneq M_1 \subsetneq M_2 \subsetneq \ldots.$$

The claim implies that $\mathrm{init}(M_i - 1) \neq \mathrm{init}(M_i)$, so there exists, for every $i \geq 1$, some $n_i \geq 0$ and $\lambda_i e_{f_i} \in \mathrm{init}(M_i)(R^{2n_i}) \backslash \mathrm{init}(M_{i-1})(R^{2n_i})$. Because $\preceq$ is a well partial ordering, there exists an infinite sequence $i_0 < i_1 < i_2 < \ldots$ such that

$$f_{i_0} \preceq f_{i_1} \preceq f_{i_2} \preceq \ldots$$

Since $\mathbb{K}$ is Noetherian, we can choose $m$ such that $\lambda_{i_m} = \sum_{j=0}^{m-1} c_j \lambda_{i_j}$ for $c_j \in \mathbb{K}$. For each $0 \le j \le m-1$, let $x_j \in M_{i_j}(R^{2n_{i_j}})$ such that $\mathrm{init}(x_j) = \lambda_{i_j} e_{f_{i_j}}$. By Lemma 3.9, there exists $\phi_j \in \mathrm{Hom}(R^{2i_j}, R^{2i_m})$ such that $\phi_j f_{i_j} = f_{i_m}$ and for any $f'_{i_j} < f_{i_j}$ in the same Hom set, $\phi_j f_{i_j} < f_{i_m}$.

Consider the element $X = \sum_{j=0}^{m-1} c_j \phi_j x_j$, which belongs to $M_{i_m-1}(R^{i_m})$. Then the properties in Lemma 3.9 implies that $\mathrm{init}(X) = \lambda_{i_m} e_{f_{i_m}} \notin M_{i_m-1}(R^{i_m})$, contradiction. □

Finally, we are ready to prove Theorem 3.3.

*Proof of Theorem 3.3.* By Lemma 2.5 and Theorem 3.12, it suffices to show that the inclusion functor $\Phi : \mathbf{OSI}(R) \to \mathbf{SI}(R)$ is finite (surjectivity is obvious). Fix $d, \omega$, and let $M = P_{\mathbf{SI}(R),(R^{2d},\omega)}$, it suffices to prove that the $\mathbf{OSI}(R)$-module $\Phi^* = M \circ \Phi$ is finitely generated. Recall that by Theorem 3.12, $Q_{d,\gamma}$ is finitely generated, for $\gamma$ a symplectic form on $R^{2d}$. If we fix $\gamma$ and a symplectic isomorphism $\tau : (R^{2d}, \omega) \to (R^{2d}, \gamma)$, then we get a natural transformation $Q_{d,\gamma} \to \Phi^*$, and the map

$$\bigoplus_{\gamma} \left( \bigoplus_{\tau} Q_{d,\gamma} \right) \to \Phi^*(M)$$

is surjective by Lemma 3.8. It follows then that $\Phi^*(M) = M \circ \Phi$ is finitely generated. □

In review: Lemma 3.10 $\Longrightarrow$ Lemma 3.9 $\Longrightarrow$ Theorem 3.12 $\Longrightarrow$ Theorem 3.3.

## 4. Orthogonal Forms on Finite Rings

Our goal is to construct an analog of **SI** for symmetric forms. However, unlike the symplectic case, rank alone does not determine the isometry class of a symmetric form. In this section, we recall the theory of orthogonal forms on finite rings. We shall always assume rings are unital and commutative and that 2 is a unit (this is important or otherwise things can break down).

### 4.1. Semilocal Rings and Finite Rings.

In [PS17], finite rings are considered, and we shall consider only finite rings as well. However, the literature of orthogonal forms often deals with semilocal rings, which are more general, so we'll make a small mention of them here.

**Definition 4.1.** A **local ring** is a ring with a unique maximal ideal. A ring $R$ is **semilocal** if $R/\mathrm{rad}\ R$ is Artinian.

For instance, any field is a local ring. There is an equivalent characterization of a semilocal ring (c.f. [Lam01]):

**Proposition 4.2.** *A ring $R$ is semilocal if and only if it has finitely many maximal ideals.* □

Therefore, it is clear that a finite ring is semilocal. Furthermore, if $\mathfrak{p}$ is a prime ideal in a finite ring $R$, then $R/\mathfrak{p}$ is a finite integral domain and therefore a field. This implies that $\mathfrak{p}$ is maximal. Therefore, we have the following result (c.f. [Lam01]):

**Proposition 4.3.** *A finite ring $R$ is the direct product of finite local rings.* □

Therefore, given a finite commutative ring $R$, we can express it as the product of finite local rings $R = \prod_{i=1}^{n} R_i$. Then, since each $R_i$ is local, there is a unique maximal ideal $\mathfrak{m}_i$ in each $R_i$, and this gives a projection map $\pi_i : R_i \to R_i/\mathfrak{m}_i$, which will be a field; in particular, the product map $\pi = \prod_{i=1}^{n}$ gives a projection map from $R$ to a product of finite fields $R/\mathfrak{m}$, where $\mathfrak{m} = \prod_{i=1}^{n} \mathfrak{m}_i$.

4.2. **Symmetric Bilinear Forms.** We now wish to define and characterize symmetric bilinear forms.

**Definition 4.4.** Let $R$ be a semilocal ring, and let $V$ be a finite-rank free $R$-module. A bilinear form $B : V \times V \to R$ is **symmetric** or **orthogonal** if $B(v, w) = B(w, v)$ for all $v, w$. The form is said to be **non-degenerate** if it it induces an isomorphism to the dual space $V^* = \mathrm{Hom}_R(V, R)$. If $B$ is non-degenerate, call the pair $(V, B)$ an **orthogonal module**. If $(V, B_V)$ and $(W, B_W)$ are two orthogonal modules, an $R$-module homomorphism $\phi : V \to W$ is called an **isometry** if $B_V(v, w) = B_W(\phi(v), \phi(w))$ for all $v, w \in V$. It is necessarily injective.

The classification of orthogonal modules over a finite ring up to bijective isometry is trickier than the symplectic case. We first have the following diagonalization theorem (c.f. [Bae06]):

**Theorem 4.5.** *Let $R$ be a semilocal ring, and let $V$ be an orthogonal $R$-module. Then, there exists a basis of $V$ in which the matrix of $B$ is diagonal and whose diagonal entries are units in $R$.* □

In other words, we can find a bijective isometry from $(V, B)$ to $(R^{\mathrm{rk}\,V}, D)$, where $D$ is a diagonal form as in the theorem. However, while this theorem greatly simplifies the classification problem, it it still redundant (for instance, permuting basis vectors in $R^{\mathrm{rk}\,V}$ will change $D$ but the resulting module is still isometric). In the case $R$ is a finite field, the answer is well-known (though a proof is hard to find, c.f. [Gla05]):

**Theorem 4.6.** *Let $\mathbb{F}$ be a finite field (of characteristic $p > 2$), and let $(V, B)$ be an orthogonal $\mathbb{F}$-module (i.e. a finite-dimensional vector space endowed with a non-degenerate symmetric bilinear form). Then, there exists a basis of $V$ such that matrix of $B$ is either 1) the identity matrix, or 2) the diagonal matrix diag$(1, \ldots, 1, x)$, where $x$ is any nonsquare in $\mathbb{F}^\times$, where different choices of $x$ yield isometric forms.*

In other words, there are two isomorphism classes, and the dimension of $V$ and the determinant of $B$ determine the isomorphism class. We can say similar result for a finite local ring. If $R$ is a finite local ring, let $\pi : R \to \mathbb{F}$ denote the projection onto its residue field $\mathbb{F} = R/\mathfrak{m}$, where $\mathfrak{m}$ is the maximal ideal in $R$.

**Theorem 4.7.** *Let $R$ be a finite local ring (where 2 is a unit), and let $(V, B)$ be an orthogonal $R$-module. Then, there exists a basis of $V$ such that matrix of $B$ is either 1) the identity matrix, or 2) the diagonal matrix diag$(1, \ldots, 1, x)$, where $x \in R$ is such that $\pi(x)$ is a nonsquare in $\mathbb{F}^\times$, and where different choices of $x$ yield isometric forms.*

*Proof.* First of all, since $R$ is a local ring, $\mathfrak{m}$ consists of the non-units in $R$, so for any unit $u \in R$, the coset $u + \mathfrak{m}$ consists solely of units. By Theorem 4.5 and applying Theorem 4.6 to the induced orthogonal $\mathbb{F}$-module, we can find a basis $\{v_1, \ldots, v_n\}$ of $V$ such that with respect to this basis the form is diagonal, $B(v_i, v_i) = (1 + t_i)^{-1}$ where $t_i \in \mathfrak{m}$ for each

$1 \leq i \leq n-1$ and $B$ satisfies one of the following two cases: either $B(v_n, v_n) = (1+t_n)^{-1}$ or $B(v_n, v_n) = (x + t_n)^{-1}$, where $t_n \in \mathfrak{m}$ and $x$ is a unit in $R$ such that $\pi(x)$ is a nonsquare in $\mathbb{F}^\times$.

Let's do case 1) first. For each $i$ in $\{1, \dots, n\}$, consider the following quadratic equation in $m$: $(1+m)^2 = 1 + t_i$, which can be rewritten as $m^2 + 2m - t_i = 0$ Since $t_i \in \mathfrak{m}$, reducing this monic polynomial modulo $\mathfrak{m}$ gives a monic quadratic equation with two distinct roots $m(m+2) = 0$, one of them being $m = 0$. By Theorem 3.12 in [GM73], it follows $m^2 + 2m - t_i = 0$ has a root $m_i$ in $\mathfrak{m}$. Then, in the basis $\{(1+m_1)v_1, \dots, (1+m_n)v_n\}$, we have $B((1+m_i)v_i, (1+m_i)v_i) = (1+m_i)^2 B(v_i, v_i) = (1+m_i)^2(1+t_i)^{-1} = 1$ for all $i$.

For case 2), we can do the same thing for $1 \leq i \leq n-1$. For $i = n$, we consider the polynomial equation $(x+m)^2 = x(x+t_n)$, which when reduced modulo $\mathfrak{m}$ gives $m(m + 2\pi(x)) = 0$. The same reasoning gives a root $m = m_n \in \mathfrak{m}$ of $(x+m)^2 = x(x+t_n)$. Then, we have $B((x+m_n)v_n, (x+m_n)v_n) = (x+m_n)^2(x+t_n)^{-1} = x$. This proves the theorem. ☐

**Corollary 4.8.** *Let $R$ be a finite ring (where 2 is a unit), and write $R = \prod_{i=1}^n R_i$ as the product of finite local rings. Then, there are $2^n$ isomorphism classes of orthogonal $R$-modules.*

*Proof.* Such a decomposition exists by Proposition 4.3. Let $e_i = (0, \dots, 0, 1_{R_i}, 0, \dots, 0)$ (nonzero in the $i$-th spot) be the central idempotent arising from $R_i = e_i R$, so $R = \bigoplus_{i=1}^n e_i R$. Then, since $1_R = e_1 + \cdots e_n$, it is clear that a bilinear form on $R$ splits as the direct sum of bilinear forms on $R_i$. Then, apply Theorem 4.7. ☐

## 5. The Category **OrI**

Fix a finite ring $R$.

**Definition 5.1.** Define the category $\mathbf{OrI}(R)$ such that the objects are orthogonal $R$-modules $(V, \beta)$, and the morphisms are isometries $\phi : (V, \beta) \to (W, \beta')$. (Here $\beta$ belongs to the first isomorphism class.)

Define the category $\mathbf{OOrI'}(R)$ such that the objects are orthogonal $R$-modules $(R^n, \beta)$, and the morphisms are row-adapted isometries $\phi : (R^n, \beta) \to (R^{n'}, \beta')$. (Here $\beta$ belongs to the first isomorphism class.)

Define the category $\mathbf{OOrI}(R)$ be the full subcategory of $\mathbf{OrI}(R)$ spanned by the orthogonal $R$-modules $(R^n, \beta_{\text{std}})$, where $\beta_{\text{std}}$ has the identity matrix with respect to the standard basis on $R^n$.

**Theorem 5.2.** *Let $R$ be a finite ring. The category of $\mathbf{OrI}(R)$-modules is locally Noetherian.*

**Theorem 5.3.** *Let $R$ be a finite ring. The category of $\mathbf{OOrI}(R)$-modules is locally Noetherian.*

Let $\Phi : \mathbf{OOrI}(R) \to \mathbf{OrI}(R)$ be the inclusion functor, and let $M = P_{\mathbf{OrI}(R),(R^d,\beta)}$ be the representable $\mathbf{OrI}(R)$-module. By Lemma 2.5 and Theorem 5.3, it suffices to prove that $\Phi$ is finite, i.e. the $\mathbf{OOrI}(R)$-module sending $(R^n, \beta_{\text{std}})$ to

$$\mathbb{K}[\mathrm{Hom}_{\mathbf{OrI}(R)}((R^d, \beta), (R^n, \beta_{\text{std}}))]$$

is finitely generated.

**Lemma 5.4.** *Let $f \in \mathrm{Hom}_{\mathbf{OrI}(R)}((R^n, \beta), (R^{n'}, \beta'))$, then we can uniquely write $f = f_1 f_2$ such that $f_2 : (R^n, \beta) \to (R^n, \gamma)$ is an isomorphism for some symmetric form $\gamma$ on $R^{2n}$, and $f_1 \in \mathrm{Hom}_{\mathbf{OOrI'}(R)}((R^n, \gamma) \to (R^{n'}, \beta'))$.*

*Proof.* Transposing, applying Lemma 3.6, then transposing back, we can uniquely write $f = f_1 f_2$, where $f_1 : R^n \to R^{n'}$ is row-adapted and $f_2 : R^n \to R^n$ is an isomorphism. We can then uniquely choose $\gamma$ a symmetric form on $R^n$ so that $f_1$ and $f_2$ are isometries. ☐

Fixing a symmetric form $\gamma$ on $R^d$ and an isomorphism $\tau \in \mathrm{Iso}_{\mathbf{SI}(R)}((R^d,\beta),(R^d,\gamma))$, we get a natural transformation between $\mathbf{OOrI}(R)$-modules $Q_{d,\gamma}$ and $\Phi^*$: at each $(R^n, \beta_{\mathrm{std}})$, let

$$\mathbb{K}[\mathrm{Hom}_{\mathbf{OOrI'}}((R^d,\gamma),(R^n,\beta_{\mathrm{std}}))] \to \mathbb{K}[\mathrm{Hom}_{\mathbf{OrI}}((R^d,\beta),(R^n,\beta_{\mathrm{std}}))]$$
$$\phi \mapsto \phi \circ \tau$$

Thus, we have the following map

$$\bigoplus_\gamma \bigoplus_\tau Q_{d,\gamma} \to \Phi^*.$$

which is surjective because of Lemma 5.4. Because each $Q_{d,\gamma}$ is finitely generated, we conclude that $\Phi^*$ is finitely generated as well.

It remains to prove Theorem 5.3, which can be proven by adapting the proof of Theorems 3.12 and 3.9 to the **OrI** case. Namely,

**Lemma 5.5.** *Fix $R, d, \omega$. There exists a well partial ordering $\preceq$ on $\mathcal{P}_R(d,\omega)$ that can be extended to a total ordering $\leq$ such that for $f, g \in \mathcal{P}_R(d,\omega)$, mapping to $R^{2n}, R^{2n'}$ respectively, satisfying $f \preceq g$, there exists some $\phi \in \mathrm{Hom}_{\mathbf{OSI}(R)}((R^{2n},\omega_{\mathrm{std}}),(R^{2n'},\omega_{\mathrm{std}}))$ such that:*

- *$g = \phi f$;*
- *For any $f_1 \in \mathrm{Hom}_{\mathbf{OSI'}(R)}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$ with $f_1 < f$, $\phi f_1 < g$.*

**Lemma 5.6.** *Fix $R, d, \beta, f, g, n, n'$ as described in Lemma 3.9, and let the rows of $g$ be $r_1, \dots, r_{2n'}$. Suppose that $f$ can be obtained from $g$ by deleting certain rows $r_j$, $j \in J = \{2i-1, 2i\}$ where $i$ ranges over some subset of $[n']$ such that $J \cap S_r(g) = \varnothing$. Then there exists $\phi \in \mathrm{Hom}_{\mathbf{OSI}(R)}(R^{2n}, R^{2n'})$ such that:*

- *For any $h \in \mathrm{Hom}_{\mathbf{OSI'}(R)}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$ with $S_r(h) = S_r(f)$, $\phi h$ can be obtained from $h$ by inserting the row $r_j$ in position $j$ for all $j \in J$. In particular, $g = \phi f$.*
- *For any $h \in \mathrm{Hom}_{\mathbf{OSI'}(R)}((R^{2d},\omega),(R^{2n},\omega_{\mathrm{std}}))$ with $S_r(h) < S_r(f)$ in lex order, then $S_r(\phi h) < S_r(g)$ in lex order.*

*Proof.* The desired map $\phi$ can be defined by the following $2n' \times 2n$ matrix: take a $2(n'-n) \times 2n$ matrix where the $k$th row $\widehat{r_k}$ coincides with $r_k$ only at positions in $S_r(f)$ and zeros elsewhere, and shuffle its rows with the rows of a $2n \times 2n$ identity matrix, such that the former rows occupy row indices in $J$. It is straightforward to check that the two points actually hold; the nontrivial part is to prove that $\phi$ preserves $\omega_{\mathrm{std}}$, which can be proven by carefully analyzing the columns of $f, g, \phi$ and using the assumptions that $f, g$ are symplectic maps. ☐

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