An Efficient Algorithm to Generate Grids Using a Modified Transformation Method

ABSTRACT

The relationship between points on a line segment and points in a square is displayed by the Hilbert curve, which shows the mapping between those points. A new alignment or algorithm can be formed so that one quadrant's end of the curve lines up with the next quadrant's beginning of the curve.

The new efficient and fast algorithm can be developed so that one quadrant's end of the curve lines up with the next quadrant's beginning of the curve. This procedure requires recursive thinking. In this paper, we rigorously define and suggest an alternative Hilbert curve. We define a new sequence of functions f(n):[0,1]→[0,1]^2 that converges to some surjective function.

INTRODUCTION

Many combinatorial mathematicians have studied on the patterns of SFCs(Space Filling Curves). Also many algorithms about those patterns have been developed. Peano, Cantor, and Hilbert proposed a geometric generation principle for the construction of the Space Filling Curves [4]. Many other space-filling curves have been discovered since 1900. These and related ideas, together with a detailed bibliography, may be found in the book by H. Sagan [1].

The relationship between points on a line segment and points in a square is displayed by the Hilbert curve, which shows the mapping between those points. A space-filling curve in n-dimensions is a continuous, surjective mapping from one domain to another domain. Basic building block of the curves is an open square formed by connected lines. A complex process made by the mathematicians need to be analyzed to simplify the procedure recursively converting each points on 1 dimensional domain, $I = \{t \mid 0 \le t \le 1\}$ which denotes the unit interval, to coordinate values on 2 dimensional unit square, $Q = \{(x, y) \mid 0 \le x \le 1, 0 \le y \le 1\}$. The values of the mapping points on a line segment have to be between zero and one, and can be decimals or fractions, written in *m/n* form [2-3].

Defining a function $f: I \to Q$, recursive smaller version of the original open square on 2 dimensional domain can fill out the whole square. The relationship between points on a line segment and points in a square is displayed by the SFCs, which shows the mapping between those points. Based on the studies of the SFCs, a new efficient alignment or algorithm can be formed so that one quadrant's end of the curve lines up with the next quadrant's beginning of the curve? If this can come true, we can reduce operational counts that cost calculating the complex mathematical and computational calculations on matrix manipulations.

Based on the building blocks using the existing SFC method, such as Peano and Hilbert curve, filling the open square can be possible by connecting lines. Using a new algorithm studied in this paper, we aim or hypothesize that complex mathematical and computational artistic patterns can be created. Using the presented SFC(Space Filling Curve) procedure recursively, smaller version of the original open square on 2 dimensional domain can finally fill out the whole square in efficient manner.

OBJECTIVES/HYPOTHESIS

The objective of this research project is developing a new algorithm that can be efficiently used in Space Filling Curve, which can be applied to image processing (compression), engineering, and design. We assume there are modified method to construct SFC using simpler and more efficient algorithm.

RESULTS/DATA

Theories

$$\mathbf{h}_0\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}\begin{bmatrix}0 & 1\\1 & 0\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\0\end{pmatrix} = \frac{1}{2}H_0\begin{pmatrix}x\\y\end{pmatrix} + \frac{1}{2}h_0 \quad (1)$$

$$\mathbf{h}_1\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}0\\1\end{pmatrix} = \frac{1}{2}H_1\begin{pmatrix}x_1\\x_2\end{pmatrix} + \frac{1}{2}h_1 \quad (2)$$

$$\mathbf{h}_2\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}1\\1\end{pmatrix} = \frac{1}{2}H_2\begin{pmatrix}x_1\\x_2\end{pmatrix} + \frac{1}{2}h_2 \quad (3)$$

$$\mathbf{h}_3\begin{pmatrix}x\\y\end{pmatrix} = \frac{1}{2}\begin{bmatrix}0 & -1\\-1 & 0\end{bmatrix}\begin{pmatrix}x\\y\end{pmatrix} + \frac{1}{2}\begin{pmatrix}2\\1\end{pmatrix} = \frac{1}{2}H_3\begin{pmatrix}x_1\\x_2\end{pmatrix} + \frac{1}{2}h_3 \quad (4)$$

$$h_j z \triangleq \frac{1}{2}H_j + \frac{1}{2}h_j, \quad j = 0, 1, 2, 3$$

where $H_0 z = \bar{z}i, H_1 z = z, H_2 z = z, H_3 z = -\bar{z}i, h_0 = 0, h_1 = i, h_2 = 1 + i$, and $h_3 = 2 + i$.

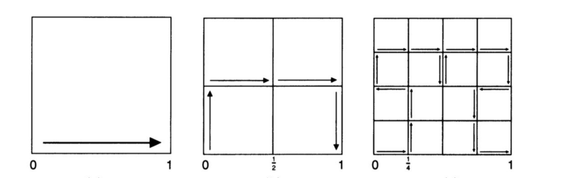


Figure 1. Recursive iteration to construct Hilbert curve

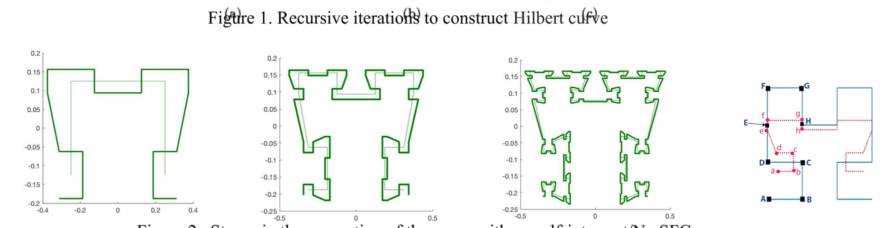


Figure 2. Stages in the generation of the curve with no self-intersect/No SFC

Space Filling Curve- Hilbert Curve

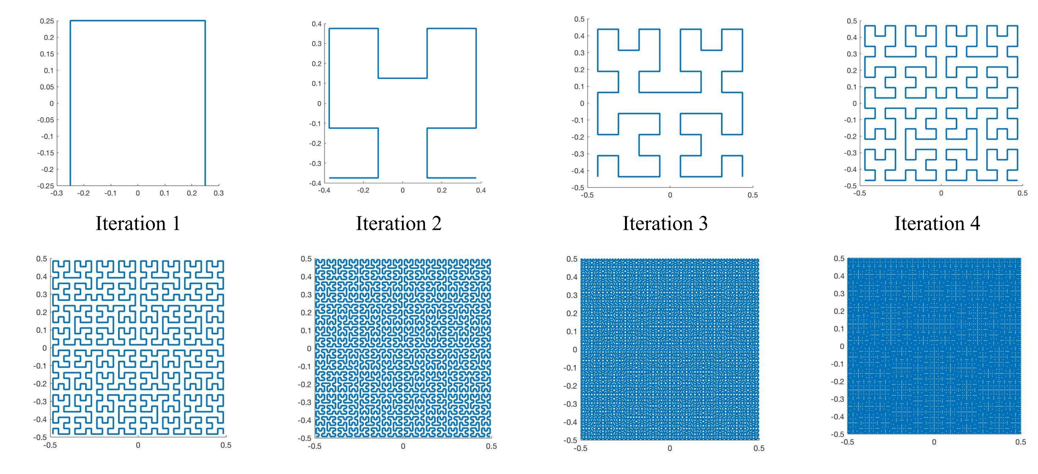


Figure 3. First eight stages in the generation of Hilbert's Space Filling Curve

The lines of each of the small squares are then converted to even smaller squares, and so on. After second iterations, we can have four smaller separate squares, and the vertices of each square are connected to form the next pattern is formed by a single continuous line.

HFC (Hilbert's space-Filling Curve) Analogues - A

Case 1

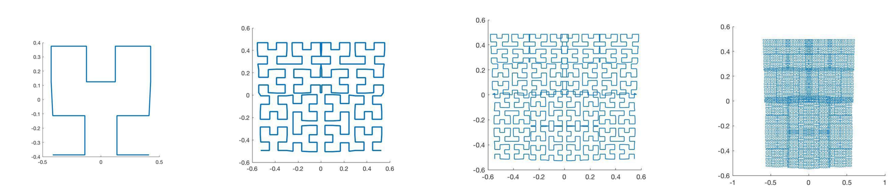


Figure 4. Stages in the generation of the curve with no slight self-intersect/incomplete SFC

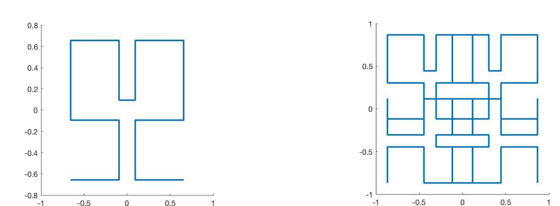
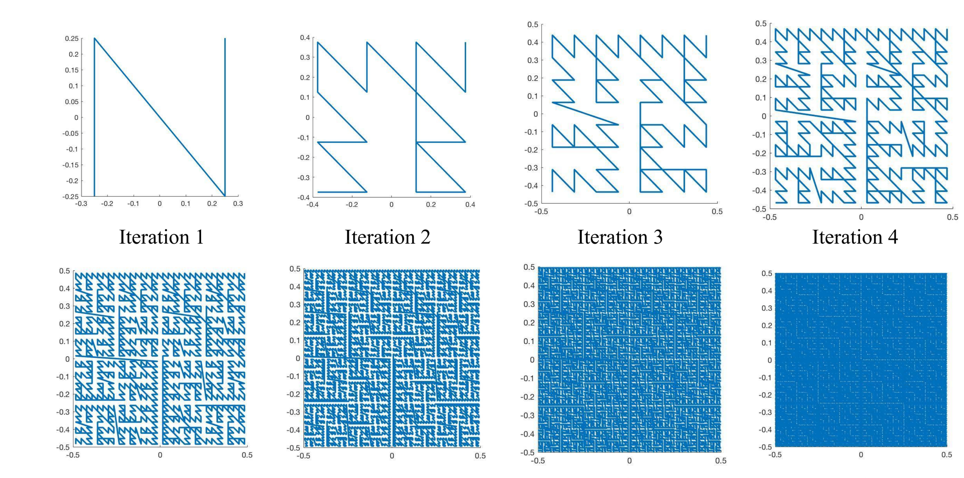


Figure 5. HFC (Hilbert's space-Filling Curve) analogue: self-intersecting curve

Case 2

Figure 6. First eight stages in the generation of space-filling curve for the N-shaped geometry



HFC (Hilbert's space-Filling Curve) Analogues – B / Case 3

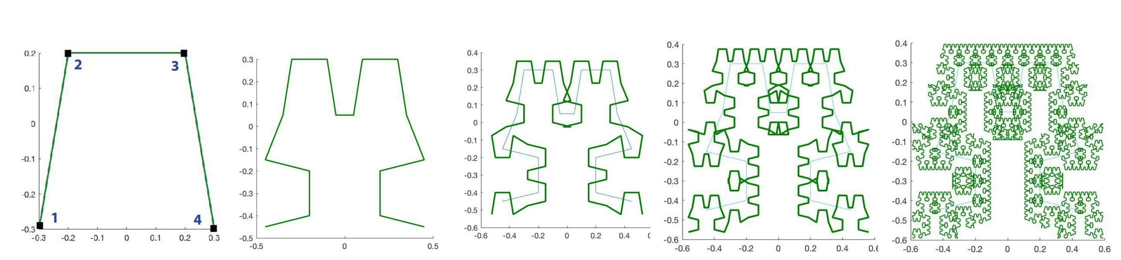


Figure 7. The 5 stages in the generation of HFC analogue (Self-intersecting)

Table 3. Data for the basic building block of the Hilbert curve (point 1-16) (Case 4B)

Coord.	1	2	3	4	5	6	7	8
x	-0.45	-0.2	-0.2	-0.45	-0.35	-0.3	-0.1	-0.05
y	-0.45	-0.4	-0.2	-0.15	0.05	0.3	0.3	0.05

Coord.	9	10	11	12	13	14	15	16
x	0.05	0.1	0.3	0.35	0.45	0.2	0.2	0.45
y	0.05	0.3	0.3	0.05	-0.15	-0.2	-0.4	-0.45

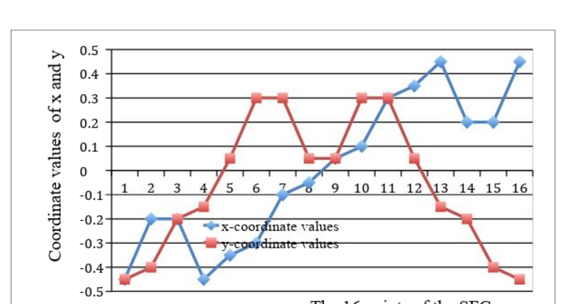


Figure 12. Coordinates of x and y of the 16 points after iteration 1 (Case 4B)

```
function [x,y] = hilbert(n)
%HILBERT Hilbert curve.
%
% [x,y]=hilbert(n) gives the
vector coordinates of points
%   in n-th order Hilbert curve
of area 1.
%
% Example: plot of 5-th order
curve
%
% [x,y]=hilbert(5);line(x,y)

if n<=0
  x=0;
  y=0;
else
  [xo,yo]=hilbert(n-1);
  x=0.5^2*[-.5^2+yo -.5^2+xo
.5^2+xo  .5^2-yo];
  y=0.5^2*[-.5^2+xo  .5^2+yo
.5^2+yo -.5^2-xo];
end
```

Applications – Design, Meshing, Math, Engineering, etc

HFC (Hilbert's space-Filling Curve) Analogue: Using Closed Square

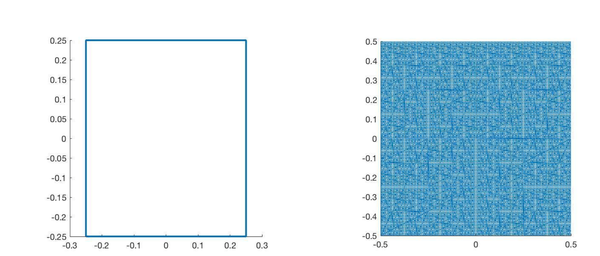


Figure 8. HFC (Hilbert's space-Filling Curve) analogue with the closed square-rectangle

Figure 8 below shows the patterns with the basic building block of the presented closed square (rectangle) formed by 4 connected lines as a starter.

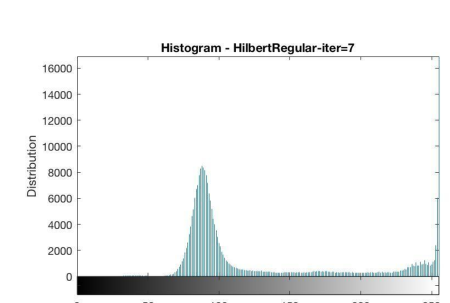
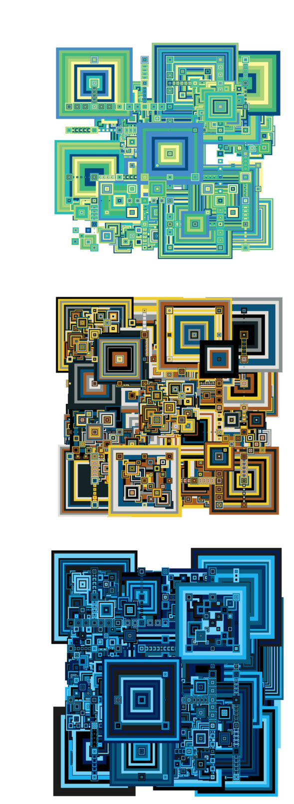


Figure 9. Histogram for the Hilbert-2 (7) with the closed square-rectangle

This stage Hilbert-2 (7) shows a level with mostly dark gray (the mean value is about 90). As the iteration increases, the mean value will decrease. The horizontal axis of the graph represents the tonal variations.



RESULTS/DATA

HFC Analogues – Case 4

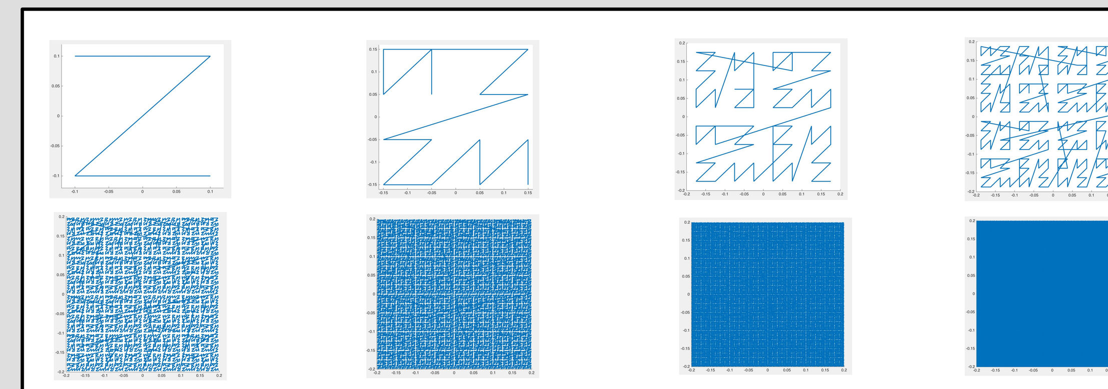


Figure 10. First eight stages in the generation of space filling curve for the Z-shaped Geometry (most efficient!)

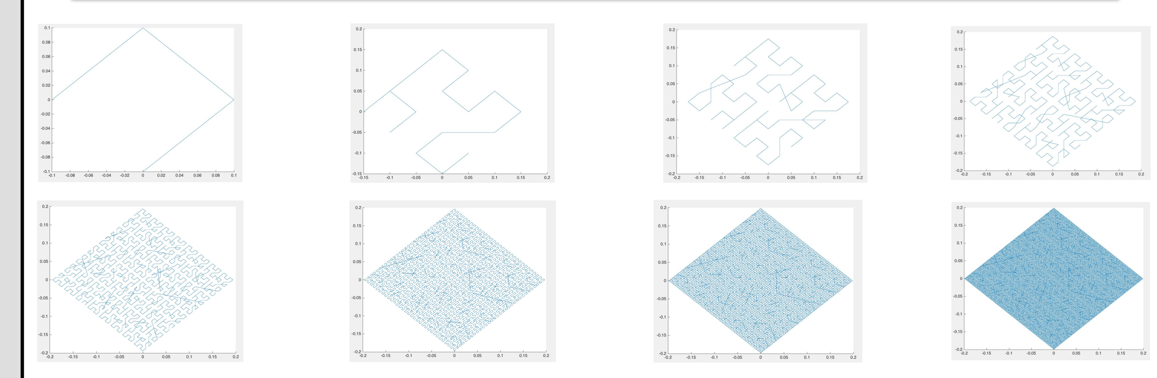


Figure 11. First eight stages in the generation of space filling curve in rhombus shape

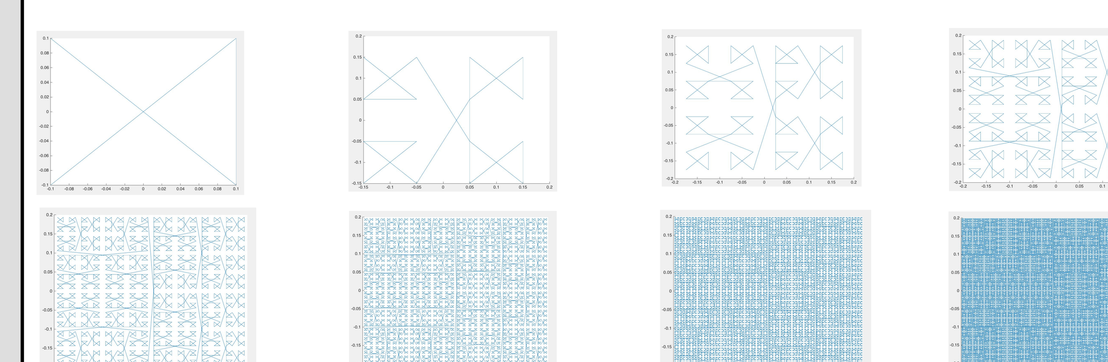


Figure 12. First eight stages in the generation of space filling curve using diagonal line shape

DISCUSSIONS

Based on the building blocks using the existing SFC methods, filling the open square can be possible by connecting lines. Using an alternative algorithm studied in this paper, we aimed that complex mathematical and computational artistic patterns can be created.

Compared to the building blocks created by the existing SFC methods, such as Peano and Hilbert curve, present research showed that filling the 2D domain was available in different manners and with simpler operations with less computational calculations. Using simplified mathematical notations, less operational works such as less matrix multiplications were observed in this research.

CONCLUSIONS

1. We checked if our proposed curves, HFC analogues, would completely cover a rectangular region of the plane if the recursion increases. We used the control target as regular space-filling curve, the Hilbert curve that never self-intersect.
2. Modified curves forming self-intersect, SFC, no self-intersect, and no SFC are all developed and tested in this research.
3. Using the presented SFC (Space Filling Curve) procedure recursively, new geometrical starters different from the smaller version of the original open square used in HFC were proposed.
4. The objective of this research project was to develop an alternative algorithm that can be efficiently used in SFC. Using an algorithm studied in this paper, we tried to reduce operational counts that cost calculating the complex mathematical and computational calculations when creating artistic and geometrical patterns, which can be applied to the image compression and combinatorial optimization in efficient manner. The emphasis of this research was on the fast computation of space-filling curve orders.

References

1. Space-Filling Curves (Springer-Verlag), Sagan, Hans (1994), ISBN-13: 978-0387942650
2. Cannon, James W. "Group invariant Peano curves", Geometry & Topology
3. Peano, G. (1890), "Sur une courbe, qui remplit toute une aire plane",