Study on the Geometric Properties in the Cevasix Triangle

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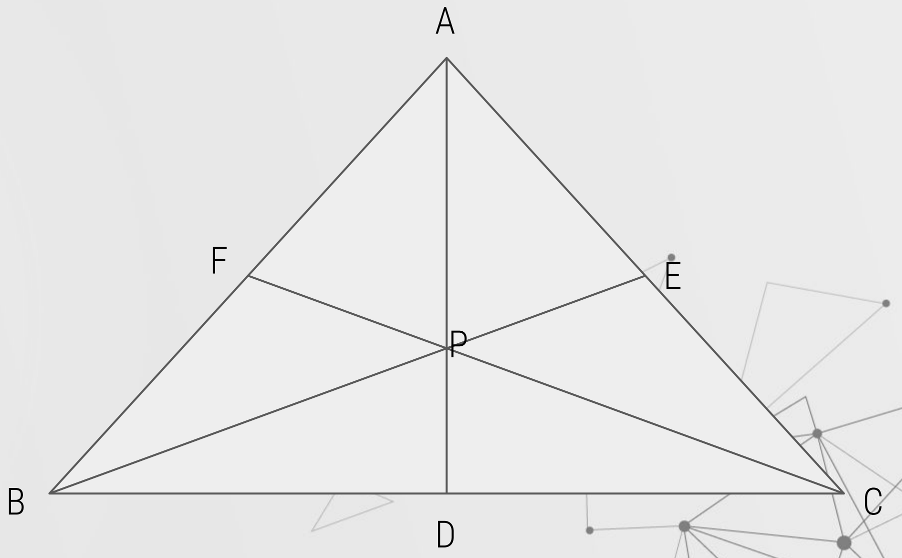
Introduction

Cevians and Stewart’s Theorem

What are Cevians?

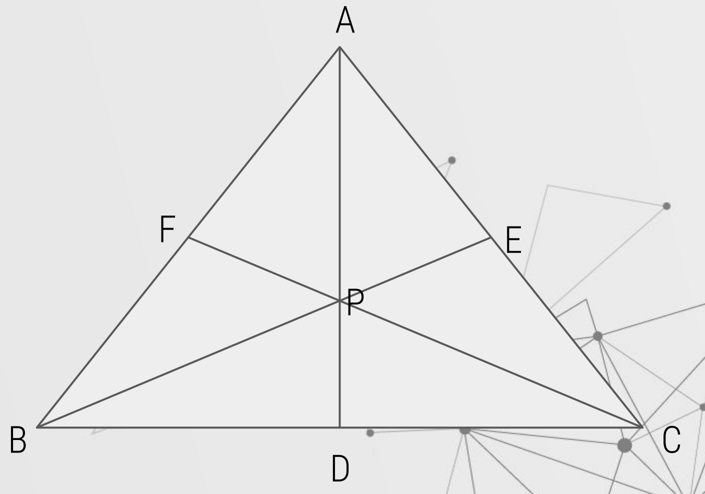
- There are 6 "ceva" triangles
  - $\triangle AEP$, $\triangle CEP$, $\triangle CDP$, $\triangle BDP$, $\triangle BFP$, and $\triangle AFP$
- The lines inside $\triangle ABC$ that form cevasix triangles = "cevians"
  - AD, CF, BE = cevians
- Main Goal: Extend the knowledge of cevians through Stewart's Theorem



Objectives

- Figure out if Stewart's Theorem can be proved using other methods such as the law of cosines and the Pythagorean theorem.

- Apply the theorem to various special cases, such as Apollonius' theorem, angle bisector theorem, isosceles triangle, and equilateral triangle, to see how the equation can expressed as the simplest form.
- Find the relationship between the perimeter formed by centroids of cevasix triangles and the perimeter of the original triangle.



Part A. Stewart's Theorem

- △ABC & a cevian from A to D on line BC are needed
- States that...
  - BD*BC*DC+ AD2 *BC= AC2*BD+ AB2*CD

- When BD = m, DC= n, BC= a, AD= d, AC = b, and AB= c,
  - man + dad = bmb + cnc

    = "A <u>man</u> and his <u>dad</u> put a <u>bomb</u> in a <u>sink</u>"

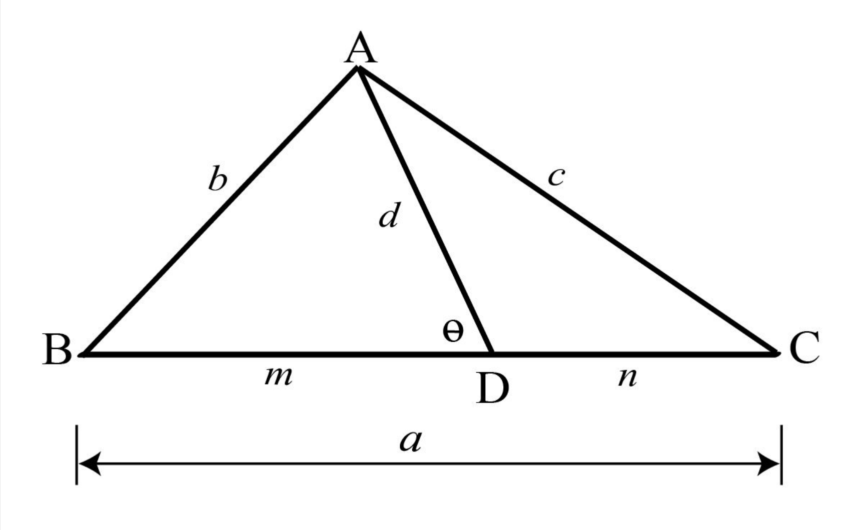


Figure 1

Alternative Methods

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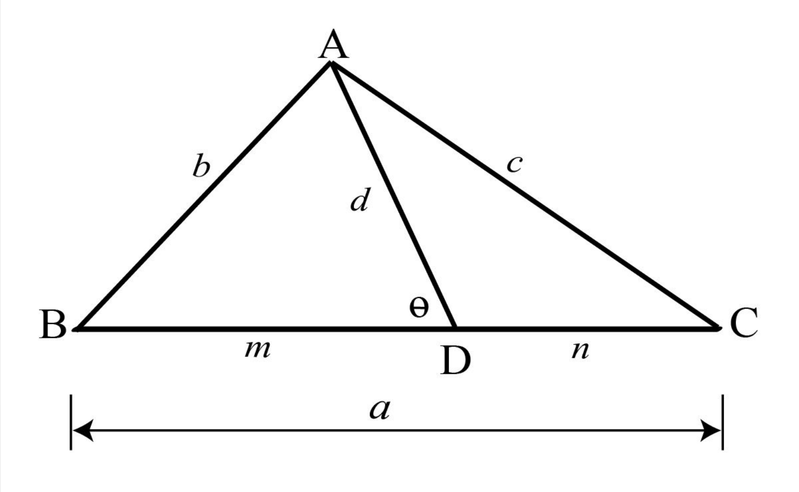
Proof I: Law of Cosines

Equations from the Law of Cosines

1) Using the Figure 1, the law of cosines could be applied to ΔABC & ΔACD
   - **Law of cosines:** $c^2=a^2 + b^2 - 2ab\cos C$
   - $c^2=m^2 + d^2-2dm\cos\theta \rightarrow$ for ΔABC
   - $b^2=d^2 + n^2 - 2dn\cos(180-\theta) \rightarrow$ for ΔACD
2) Note $\cos(180-\theta)= -\cos\theta$
   - $b^2=d^2 + n^2 + 2dn\cos\theta$
3) Multiply "n" to the first equation & "m" to the second equation
   - $c^2\mathbf{n}=m^2\mathbf{n} + d^2\mathbf{n}-2dm\mathbf{n}\cos\theta$
   - $b^2\mathbf{m}=d^2\mathbf{m}+ n^2\mathbf{m}+ 2dn\mathbf{m}\cos\theta$
4) Add two equations
   - $c^2\mathbf{n}+b^2\mathbf{m} = m^2\mathbf{n} + d^2\mathbf{n} + d^2\mathbf{m}+ n^2\mathbf{m}$
   - $c^2\mathbf{n}+b^2\mathbf{m} = mn(m+n) + d^2(m+n)$
5) Replace m+n by a
   - <u>man + dad = bmb + cnc</u>



Alternative Methods

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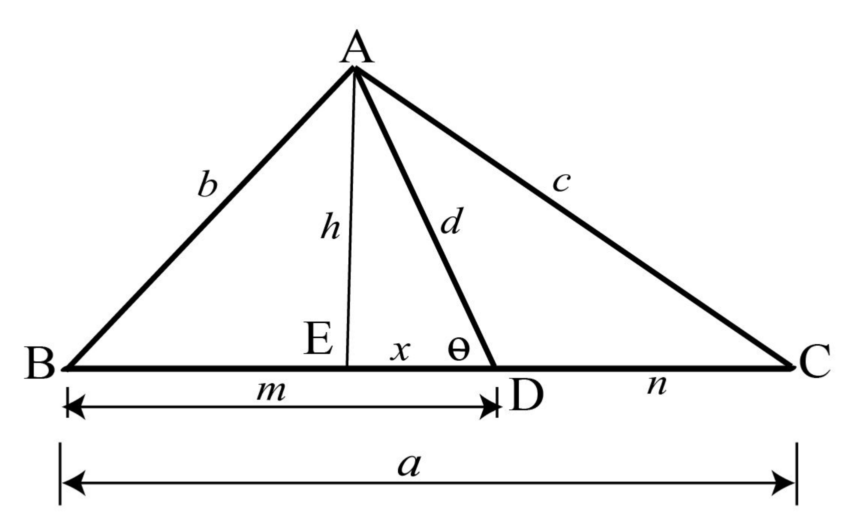
Proof II: Pythagorean Theorem

New Triangle from the Pythagorean Theorem

- Given ΔABC shown in Figure 2,
    - *h*= perpendicular line from A to BD
    - Point E = where h lands
      → ED = x

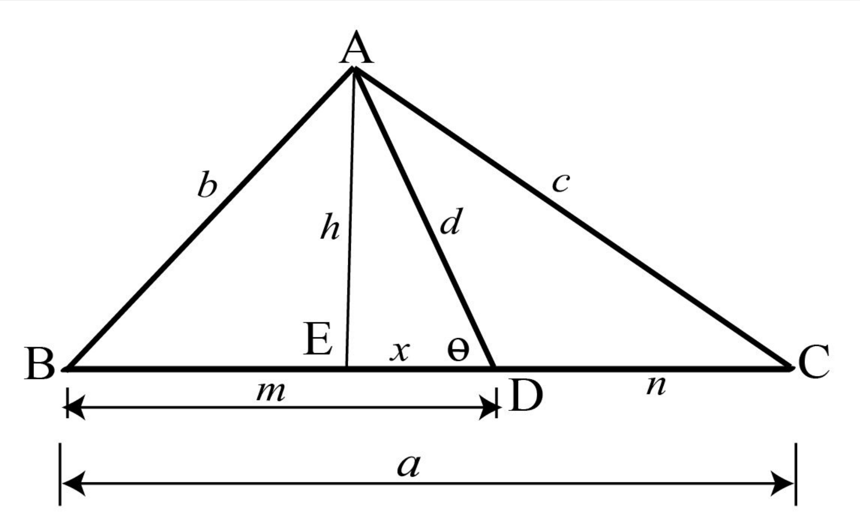


|BD|=m |DC|=n |AE|=h |ED|=x

Figure 2: Using Pythagorean Theorem

Proof Using the Pythagorean Theorem

1) According to the Pythagorean Theorem, three equations can be set up with *h* and x:
   a) $b^2 = h^2 + (x+n)^2$
   b) $d^2 = h^2 + x^2$
   c) $c^2 = h^2 + (m-x)^2$
2) Using (b), solve for x
   a) $x = \sqrt{d^2 - h^2}$
3) Substitute x to 1-(a) and 1-(c)
   a) $b^2 = h^2 + (\sqrt{d^2-h^2}+n)^2$
      $= h^2 + n^2 + (d^2 - h^2) + 2n\sqrt{d^2-h^2}$
   b) $c^2 = h^2 + (m-\sqrt{d^2-h^2})^2$
      $= h^2 + m^2 + (d^2 - h^2) - 2m\sqrt{d^2-h^2}$
4) Multiply 3-(a) by m
   a) $b^2\mathbf{m} = h^2\mathbf{m} + n^2\mathbf{m} + (d^2 - h^2)\mathbf{m} + 2\mathbf{m}n\sqrt{d^2-h^2}$
5) Multiply 3-(b) by n
   a) $c^2\mathbf{n} = h^2\mathbf{n} + m^2\mathbf{n} + (d^2 - h^2)\mathbf{n} - 2m\mathbf{n}\sqrt{d^2-h^2}$



|BD|=m |DC|=n |AE|=h |ED|=x

Figure 2: Using Pythagorean Theorem

Proof Using the Pythagorean Theorem

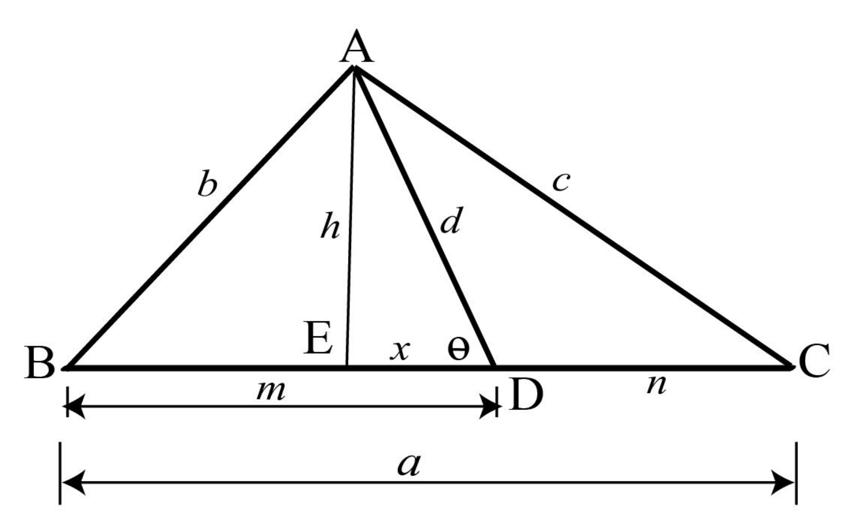
6) Add up 4-(a) and 5-(a)
   a) $b^2\mathbf{m} + c^2\mathbf{n} = h^2\mathbf{m} + n^2\mathbf{m} + (d^2-h^2)\mathbf{m} + h^2\mathbf{n} + m^2\mathbf{n} + (d^2-h^2)\mathbf{n}$

      $= h^2 (m+n) + (d^2-h^2)(m+n) + nm(m+n)$

      $= h^2 (m+n) + d^2(m+n) - h^2(m+n) + nm(m+n)$

      $= d^2(m+n) + nm(m+n)$
7) Therefore, the final answer is...
   a) $b^2m + c^2n = d^2(m+n) + nm(m+n)$

   → <u>man + dad = bmb + cnc</u>



|BD|=m |DC|=n |AE|=h |ED|=x

Figure 2: Using Pythagorean Theorem

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Proof III: Special Cases

Apollonius' Theorem, Angle Bisector, Altitude, Isosceles Triangle, Equilateral Triangle

Apollonius' Theorem

- It is a special case of Stewart's Theorem when the cevian is a median

1) Substitute m to the equation of Stewart's Theorem
   a) $m^2(2m) + d^2(2m) = bmb + cmc$
2) Divide 1-(a) by m
   a) $2m^2 + 2d^2 = b^2 + c^2$

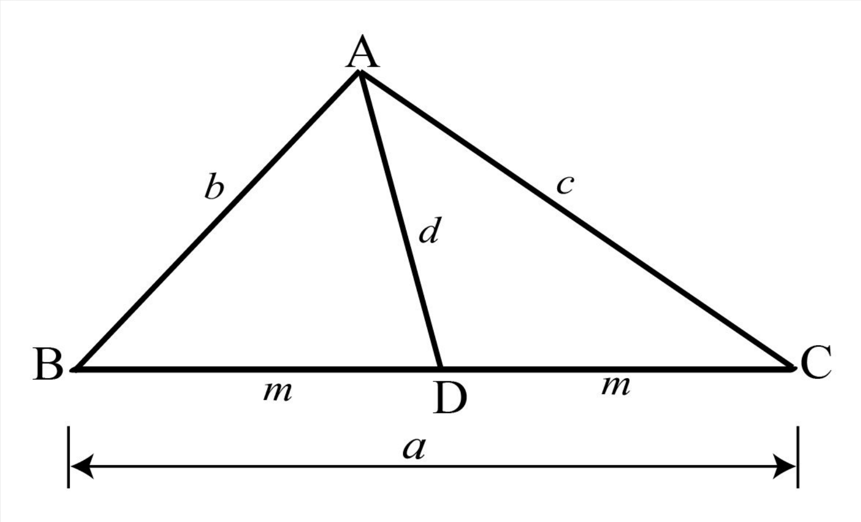

Figure 3: Apollonius' Theorem

Angle Bisector

- It is a special case of Stewart's Theorem when AD is an angle bisector

1) According to the Angle Bisector Theorem,
   a) $b/n=c/m$
      $= bm = cn$
2) Then, substitute, factor, and cancel
   a) $man + dad = (cn)b + (bm)c$
      $= a(mn+d^2)= bc(n+m)$
3) Replace m+n by a
   a) $a(mn+d^2)=bca$
4) Factor out a
   a) $mn+d^2= bc$

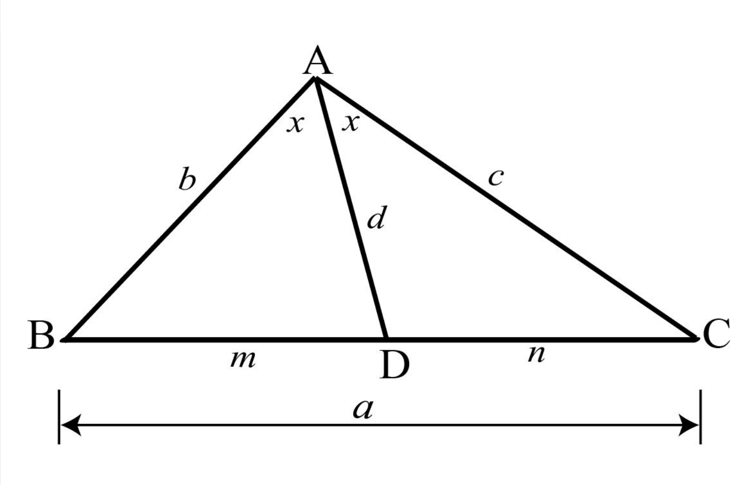

Figure 4: Angle Bisector

Altitude

- It is a special case of Stewart's Theorem when the cevian is an altitude

1) Under this condition, use the Pythagorean Theorem
   a) $c^2=m^2+d^2$
   b) $b^2=n^2+d^2$
   c) $man + d^2a = b^2m+c^2n$
2) Substitute 1-(a) and (b) to (c)
   a) $man + d^2a = (n^2+d^2)m+(m^2+d^2)n$

      $= n^2m + d^2m + m^2n + d^2n$

      $= d^2(m+n) +mn (m+n)$
3) m+n is merely "a"--> satisfies Stewart's Theorem (when cevian is an altitude)

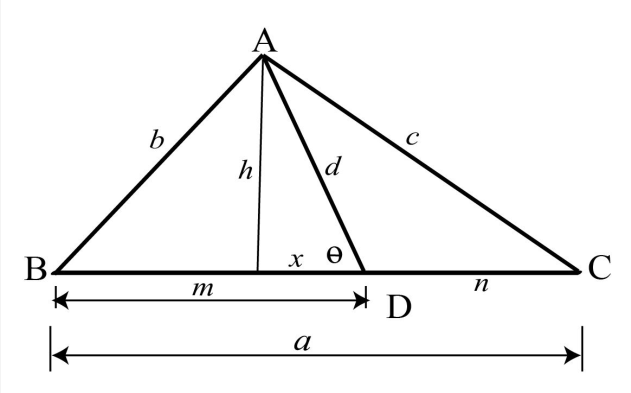

Figure 4-1: Altitude

Isosceles Triangle

- If the triangle is isosceles, with the cevian intersecting the base, then both sides are labeled as length *b*
  - $man + dad = bmb + bnb \rightarrow man + dad = b^2\,(m+n)$

1) Factor out a
   a) $a(mn + d^2) = b^2\,(m+n)$
2) Replace m+n by “a”
   a) $a(mn + d^2) = b^2 a$
3) Cancel out “a”
   a) $mn + d^2 = b^2$

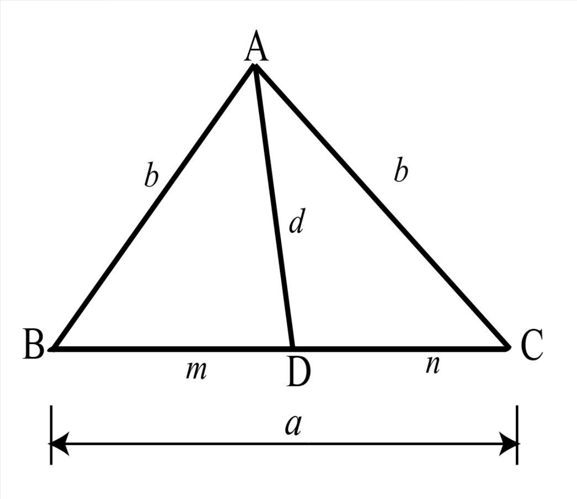

Figure 4-2: Isosceles Triangle

Equilateral Triangle

- It is a special case of isosceles triangle but with all three sides as a length *b*
- Therefore, the equation will be
    - $mb\text{-}m+d^2=b^2$

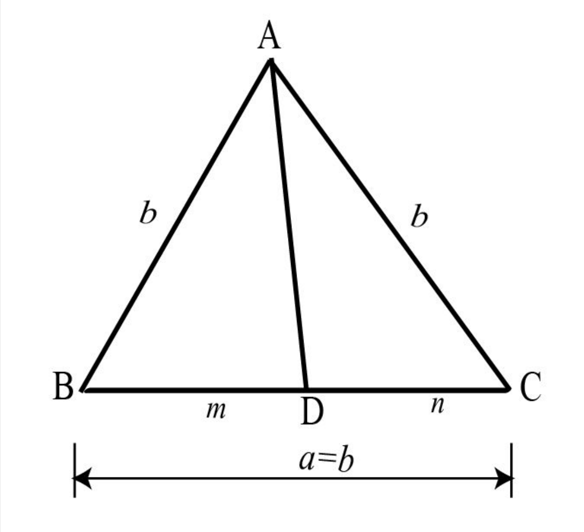

Figure 4-3: Equilateral Triangle

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Extension

Two Cevians from the Same Vertex, Two Intersecting Cevians

Two Cevians from the Same Vertex

1) Apply Stewart's Theorem to ΔABE and ΔACB with AD & AE as cevians, respectively. Note that AB is now b instead of c & AC is c instead of b
   a) $mn(m+n) + d^2(m+n) = e^2m + b^2n \rightarrow \Delta ABE$
   b) $(m+n)ao + e^2a = c^2(m+n) + b^2o \rightarrow \Delta ACB$

2) By using 1- (a), solve for m+n
   a) $m+n=(b^2n+e^2m) / (mn+d^2)$

3) For 1-(b), shift around the terms & factor them out
   a) $e^2a-b^2o=(m+n)(c^2-oa)$

4) Substitute $(b^2n+e^2m) / (mn+d^2)$ for m+n into 3-(a) & cross multiply
   a) $(e^2a-b^2o)(mn+d^2)=(b^2n+e^2m)(c^2-oa)$

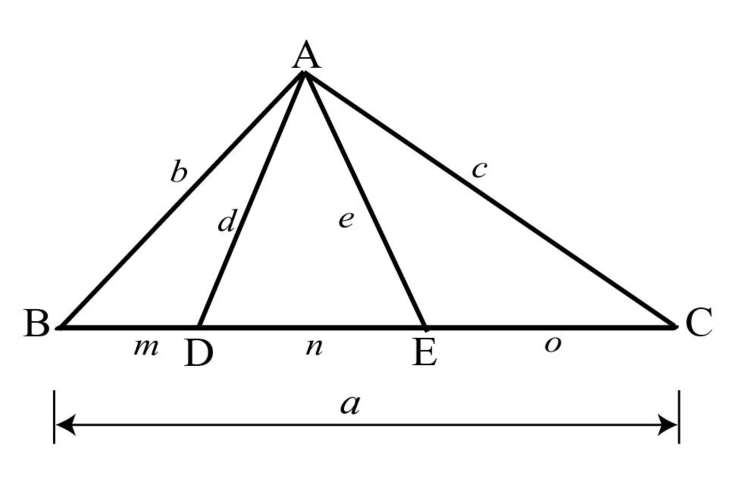


Figure 5: Two Cevians From the Same Vertex

Two Cevians from the Same Vertex

5) Expand & Factor

a) $e^2amn + e^2ad^2 - b^2omn - b^2od^2 = c^2b^2n - b^2noa + e^2mc^2 - e^2moa$

$= e^2amn + e^2ad^2 + e^2moa - e^2mc^2 = b^2od^2 + b^2omn + c^2b^2n + b^2noa$

$= e^2(man+dad+mao-cmc) = b^2(dod+mon+cnc-nao)$

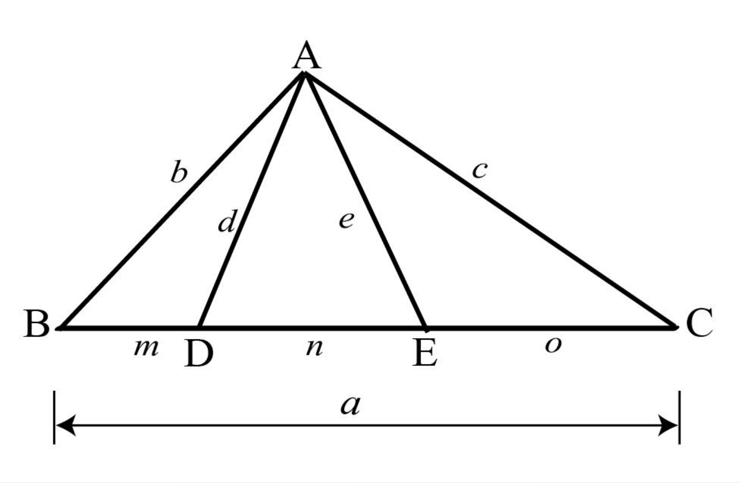


Figure 5: Two Cevians From the Same Vertex

Two Intersecting Cevians

1) By using Stewart's Theorem, 4 equations can be made:
   a) $b(b_1b_2+ e^2) = c^2b_1+a^2b_2 \rightarrow \Delta ABC$ (Cevian: BE)
   b) $c(c_1c_2 + d^2) =b^2c_2+a^2c_1 \rightarrow \Delta BCA$ (Cevian: CD)
   c) $e(e_1e_2+d^2{}_1) = a^2e_1+b^2{}_1e_2 \rightarrow \Delta BCE$ (Cevian: CF)
   d) $d(d_1d_2+e^2{}_2) = c^2{}_2d_1+a^2d_2 \rightarrow \Delta CBD$ (Cevian: BF)

2) Solve for a^2 & equal all equations to each other

$$a^2 = [b(b_1b_2+ e^2)-(c^2b_1)] / b_2$$

$$= [c(c_1c_2 + d^2)-(b^2c_2)] / c_1$$

$$= [e(e_1e_2+d^2{}_1)-(b^2{}_1e_2)] / e_1$$

$$= [d(d_1d_2+e^2{}_2)-(c^2{}_2d_1)] / d_2$$

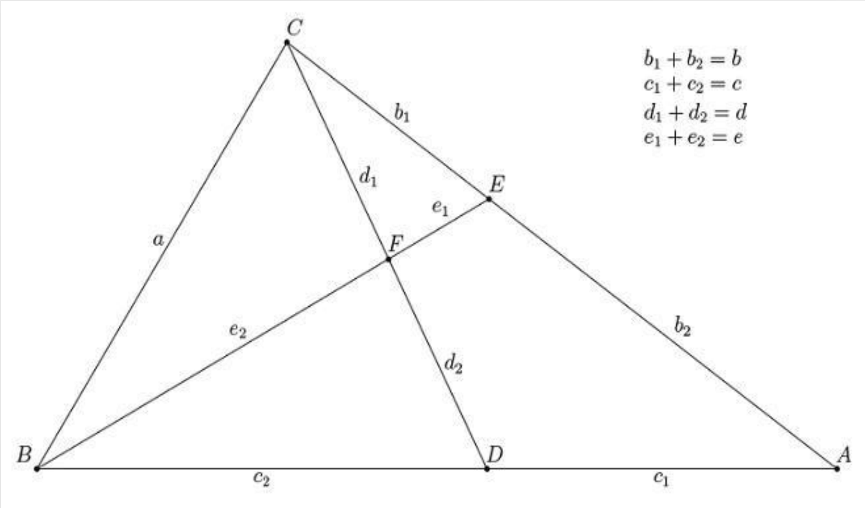


Figure 6: Two Intersecting Cevians

Trapezoid Transversal

- By using Stewart's Theorem, the relationship between transversal & the sides of a trapezoid can be found.
- See Figure 7, a transversal passes line AB and CD & AD, FG, and BC are extended to point E
  - ΔEDC and ΔEAB become similar triangles
  - Using the ratio of ΔEDC: ΔEAB = 1:x, all sides were set up

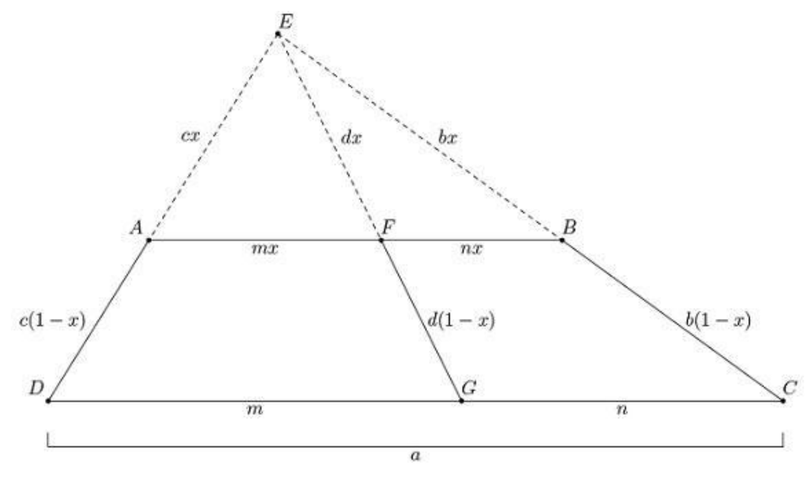


Figure 7: Trapezoid Transversal Derivation

Trapezoid Transversal

Equation man + dad = bmb + cnc is multiplied by $(1-x)^2$

$man(1-x)^2 + dad(1-x)^2 = bmb(1-x)^2 + cnc(1-x)^2$

$= m(1-x)an(1-x) + a(d(1-x))^2 = m(b(1-x))^2 + n(c(1-x))^2$ → Figure 7

$= (M-m)(N-n)a + dad = bMb + cNc$ → Figure 8

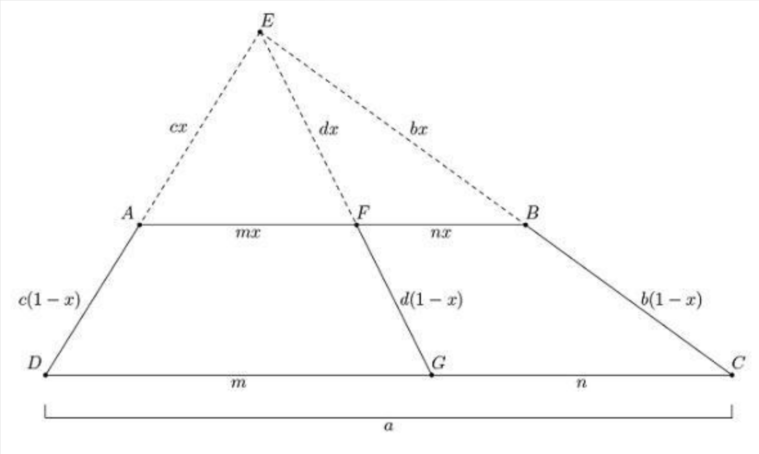


Figure 7: Trapezoid Transversal Derivation

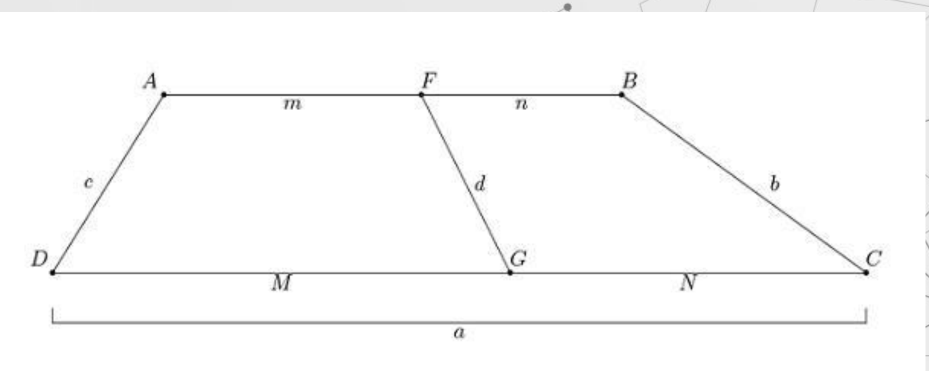


Figure 8: Trapezoid Transversal

Three Cevians from the Same Vertex

1) Stewart's Theorem is used on ΔAEC
   a) $e^2p + c^2o = (o+p)(f+op)$

2) Solve for e^2
   a) $e^2 = (of + pf + o^2p + op^2 - c^2o)/p$

3) Substitute 2-(a) to the equation from two cevians from the same vertex
   a) $e^2(man+dad+mao-cmc) = b^2(dod+mon+cnc-nao)$
   → equation from two cevians
   b) $(of + pf + o^2p + op^2 - c^2o)\,(man + dad + mao - cmo)$
   $= pb^2(dod + mon + cnc - nao)$

4) Expand & Factor
   a) $[(o+p)(f+op)n - c^2o](man + dad + mao - cmc)$
   $= pb^2(dod + mon + cnc - nao)$

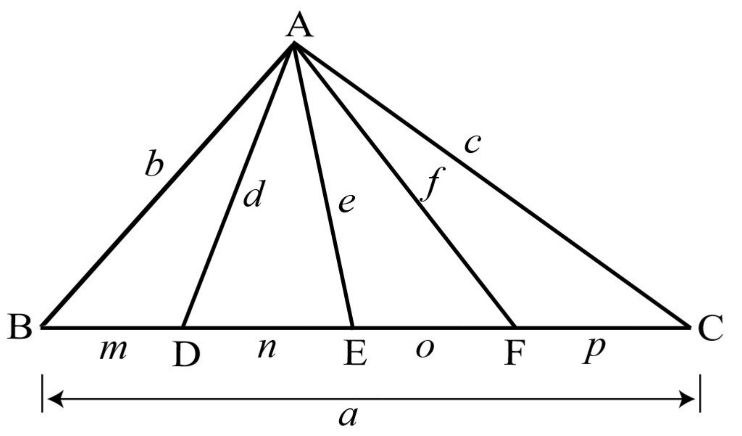
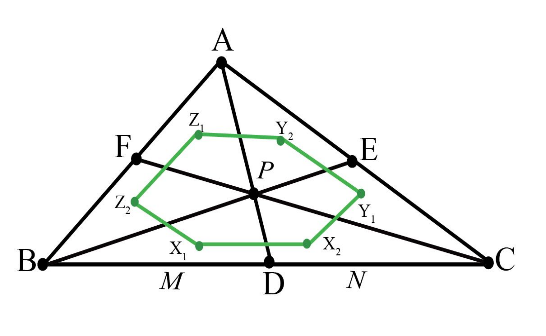


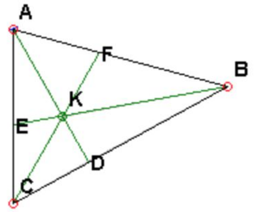
Figure 9: Three Cevians From the Same Vertex

Part B. Cevasix Triangle



Trigonometric Form of Ceva's Theorem

Ceva's theorem provides a unifying concept for several apparently unrelated results. The theorem states that, in ΔABC, three Cevians AD, BE, and CF are concurrent iff the following identity holds:



$$\frac{AF}{FB}\cdot\frac{BD}{DC}\cdot\frac{CE}{EA}=1.$$

The theorem has a less known trigonometric form

$$\boxed{\frac{\sin(\angle ABE)}{\sin(\angle CBE)}\cdot\frac{\sin(\angle BCF)}{\sin(\angle ACF)}\cdot\frac{\sin(\angle CAD)}{\sin(\angle BAD)}=1,}$$

or

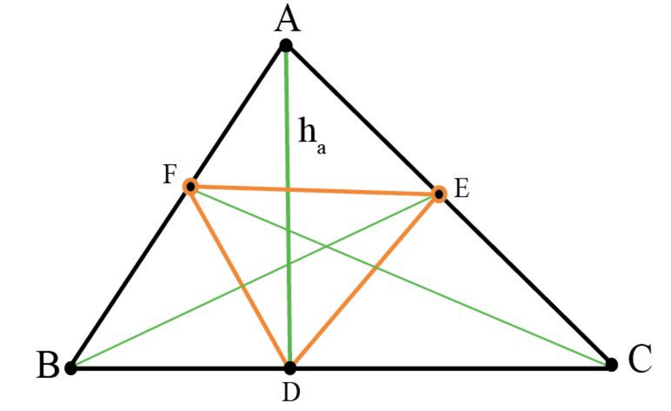
$$\boxed{\sin(\angle ABE)\cdot\sin(\angle BCF)\cdot\sin(\angle CAD)=\sin(\angle CBE)\cdot\sin(\angle ACF)\cdot\sin(\angle BAD).}$$

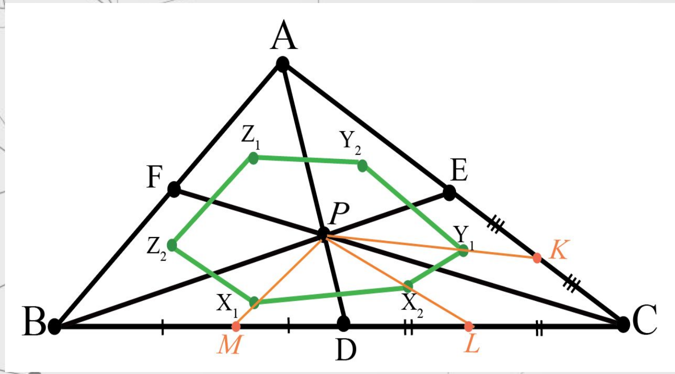
1) Let the centroids of 6 cevasix triangles X_1, X_2, Y_1, Y_2, Z_1, and Z_2.

2) Assume the semi perimeter of the hexagon X1, X2, Y1, Y2, Z1, Z2 formed by all the centroids is ***s***, the semi perimeter of ABC is ***u***, and the semi perimeter of DEF ***t***. Let the altitude from vertex A be h_a.

3) The following inequality can be proved.
   1. $3s = u+t$
   2. $3s = u+ h_a\sin(A)$

Cevasix Triangle





Assume the semi perimeter of the hexagon X1, X2, Y1, Y2, Z1, Z2 formed by all the centroids is ***s***, the semi perimeter of ABC is ***u****,* and the semi perimeter of DEF ***t***.

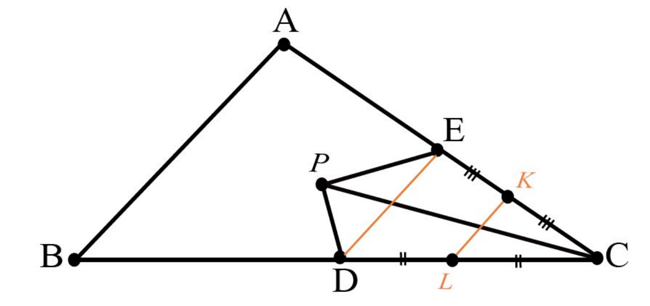
Let the altitude from vertex A be h_a.

Then the following inequality can be proved.

$3s = u+t$ or

$3s = u+ h_a \sin(A)$

Cevasix Triangle



From the similarity

[DE]//[KL] and

½ |DE| = |KL|

Thus,

$|Y_1X_2| = 2/3|KL| = (⅔)\ |DE|/2 = |DE|/3$

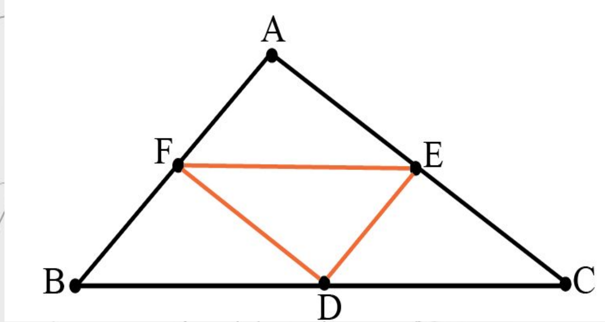
And,

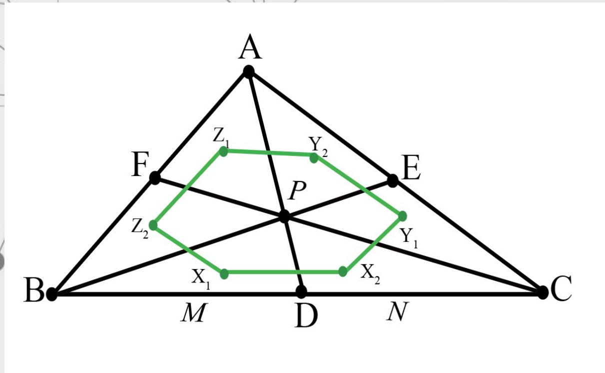
$|Y_1X_2|+|Z_1Y_2|+|X_1Z_2|$

$= ⅓(|DE|+|EF|+|FD|)$

=Perimeter(DEF)/3

Cevasix Triangle





Thus, the perimeter surrounded by X_1, X_2, Y_1, Y_2, Z_1, and Z_2 is can be found in terms of u and t.

Perimeter($X_1X_2Y_1Y_2Z_1Z_2$) = 2s

= 2u/3 + Perimeter(DEF)/3

Perimeter(DEF) = 2t

2s = 2u/3 + 2t/3

Finally,

3s = u+t

Or, if DEF is the orthic triangle of ABC,

Perimeter(DEF) = 2t =($2h_a$) sin(A)

3s = u+ $h_a$sin(A)

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Conclusion & Future Considerations

Conclusion

With the main use of Stewart's Theorem, the relationship among special cases was analyzed. This project found that the formula from Stewart's Theorem (man + dad = bmb + cnc) could be extended to more various and interesting cases such as two cevians, two intersecting cevians, and three cevians. Also a cevasix triangle was studied to find a new relation between the perimeter formed by centroids of cevasix triangles and the perimeter of the original triangle.

Summary:

- Stewart's Theorem could be modified depending on different conditions. This allowed me to conclude that various special cases, two cevians, and two intersecting cevians can contribute to the alternate equation.

- Special cases, such as Apollonius' theorem, angle bisector, altitude, isosceles triangle, and equilateral triangle, were useful because they could make the equation as the simplest form.

- It was possible to derive an equation with three cevians from the same vertex by using the equation for two cevians from the same vertex.

- The relationship between the perimeter formed by centroids of cevasix triangles and the perimeter of the original triangle was found.

Future Considerations

- We can extend Stewart's Theorem to apply to other polygons and it might be possible to relate the perimeter of the polygon to areas of triangles in the polygon.

- It will be interesting to derive formulas with n-cevians from the same vertex by using the formerly derived equation for (n-1) cevians.

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