Study on the Geometric Properties in the Cevasix Triangle

Abstract

In geometry, the relation between the cevians and the sides of a triangle can be found in Stewart's theorem. In this research, the relationship between the cevians, the sides of a triangle, the segments it forms on the opposite side, and the properties of cevasix triangles are studied. Using Stewart's Theorem and other related theories, new relationships under special circumstances were analyzed. It was possible to change the existing formula and extend them to various cases with more variables using the presented ideas in this paper.

First, cevians and their following formulas in a cevasix triangle were studied in this project. All the related formulas and equations were reviewed to find correlations in the formula. At first, this project discussed how a certain law and theorem, which we are already familiar with, can be utilized to find modified forms with the equations derived from Stewart's Theorem. Special cases and basic properties, such as in Apollonius' Theorem, angle bisector, isosceles triangle, and equilateral triangle, were used because they could make the equation in the simplest form. While two cevians and two intersecting cevians do not produce a simple equation, they explain how Stewart's theorem is versatile and can fit into various interesting cases. Finally, formulas with n-cevians from the same vertex are possible by using the derived equation for (n-1) and new correlations in the cevasix triangle.

Introduction

Matthew Stewart was a Scottish mathematician who was accepted into the University of Glasgow in 1734. There, he worked with a mathematics professor named Robert Simson. Simson introduced Stewart to the study of Greek mathematics. Later, Simson recommended Stewart to go to the University of Edinburgh to work under Mr. Colin Maclaurin. However, they continued their correspondence, and Stewart expressed his desire to study pure geometry. Simson made all his works freely available to Stewart and encouraged him to publish General Theorems in 1746. We have only recently learned that this theorem associated with Stewart was originally Simson's idea. This research revolves around Stewart's Theorem. The discussion topics include the theorem's history, two ways to prove it, special cases, and some extensions. Information on the history of Matthew Stewart, the mathematician to whom this theorem is named and credited, is provided in the history section.

Moreover, two ways of proving Stewart’s Theorem using the Law of Cosines and the Pythagorean theorem are mainly discussed. Two special cases of Stewart’s Theorem - when the cevian is a median and when it is an angle bisector- are also examined and proved by this research. The former is also known as Apollonius’ Theorem. The research also underscores this theorem’s use to derive new formulas for certain various situations. The first one referred to a triangle with two cevians from a single vertex, the second one to a triangle with two intersecting cevians, and the last one to a trapezoid with a transversal cutting through its parallel sides. In all three cases, the formulas are derived by relating the lengths of the sides in each figure. A sample contest problem that could easily be solved with Stewart’s Theorem is provided.

Finally, relationships between the perimeter formed by centroids of cevasix triangles and the perimeter of the original triangle were found.

Stewart's Theorem

Stewart's Theorem states that given any ΔABC and a cevian from vertex A to a point D on $\overline{BC}$, $BD * BC * DC + AD^2 * BC = AC^2 * BD + AB^2 * CD$. This theorem is remembered using a mnemonic device: $man + dad = bmb + cnc$ or “A man and his dad put a bomb in the sink," where $BD = m$, $DC = n$, $BC = a$, $AD = d$, $AC = c$, and $AB = c$bb

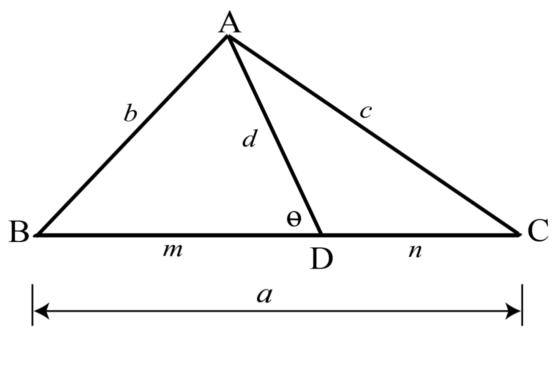


Figure 1: Stewart’s triangle

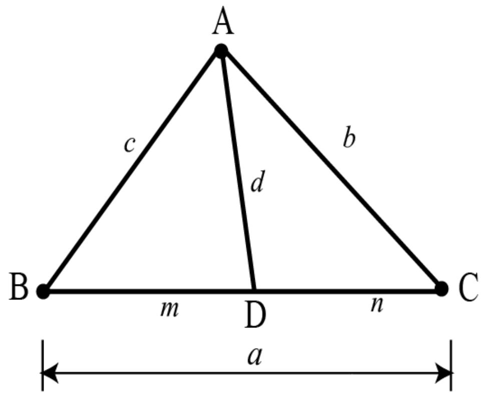


Figure 2: Studied triangle 1

Proof Using Law of Cosines

A common way to prove Stewart's Theorem is by the Law of Cosines. Referring to Figure 1, the Law of Cosines could be applied twice to ΔABC and ΔACD. We now have

$$c^2 = m^2 + d^2 - 2dmcos\,\theta \qquad (1)$$

$$b^2 = d^2 + n^2 - 2dncos\,(180 - \theta)$$

Note, however, that $cos\,(180 - \theta) = -cos\,\theta$. The second equation would then be simplified to

$$b^2 = d^2 + n^2 + 2dncos\,\theta$$

Next, the first equation could be multiplied by *n* and the second by *m* to cancel out $2dmncos\,\theta$in the later step. This change results in

$$c^2\boldsymbol{n} = m^2\boldsymbol{n} + d^2\boldsymbol{n} - 2dm\boldsymbol{n}cos\theta$$
$$b^2\boldsymbol{m} = d^2\boldsymbol{m} + n^2\boldsymbol{m} + 2d\boldsymbol{m}ncos\theta$$

When these two equations are added, $+\ 2dmncos\,\theta$ and $-\ 2dmncos\,\theta$ cancel out, and we are left with

$$c^2n + b^2m = m^2n + d^2n + d^2m + n^2m$$

$$c^2 n + b^2 m = mn(m + n) + d^2(m + n)$$

We factor and replace $m + n$ by a.

$$man + dad = bmb + cnc$$

Proof Using Pythagorean Theorem

An alternate proof uses the Pythagorean Theorem instead. Given ΔABC as seen in Figure 2, h as the height perpendicular to $\overline{BC}$ from A and a point E where h lies on `$\overline{BC}$can be made; therefore, ED= x. By using h and x, we could write three equations using the Pythagorean Theorem.

$$b^2 = h^2 + (n + x)^2 \qquad (2.1)$$

$$d^2 = h^2 + x^2 \qquad (2.2)$$

$$c^2 = h^2 + (m - x)^2 \qquad (2.3)$$

From the second equation, we solve for x:

$$x = \sqrt{d^2 - h^2} \qquad (2.4)$$

Then we substitute this into the first equation (2.1) and third equation (2.3) and simplify.

Equation (2.1) becomes

$$b^2 = h^2 + (n + x)^2$$
$$= h^2 + n^2 + x^2 + 2nx$$

And using the equation (2.4)

$$b^2 = h^2 + n^2 + (d^2 - h^2) + 2n\sqrt{d^2 - h^2} \qquad (2.5.1)$$

And the third equation (2.3) becomes

$$c^2 = h^2 + (m - x)^2$$

$$= h^2 + m^2 + x^2 + 2mx$$

And using the equation (2.4)

$$c^2 = h^2 + m^2 + (d^2 - h^2) - 2m\sqrt{d^2 - h^2} \qquad (2.5.2)$$

Multiplying both sides of the equation (2.5.1) by m

$$b^2m = h^2m + n^2m + (d^2 - h^2)m + 2mn\sqrt{d^2 - h^2} \quad (2.6)$$

Multiplying both sides of the equation (2.5.2) by n

$$c^2n = h^2n + m^2n + (d^2 - h^2)n - 2mn\sqrt{d^2 - h^2} \quad (2.7)$$

Adding up the two equations (2.6) and (2.7), $2mn\sqrt{d^2 - h^2}$ canceled out and it becomes

$$b^2m + c^2n = h^2m + n^2m + h^2n + m^2n + (d^2 - h^2)m + (d^2 - h^2)n$$

$$= h^2 (m + n) + mn (m+n) + (d^2 - h^2)(m + n)$$

$$= h^2 (m + n) + mn (m+n) + (d^2 - h^2)(m + n)$$

= h2 (m + n) + mn(m+n) + (d2 - h2)(m + n)

= h2 (m + n) +(d2 - h2)(m + n) + mn(m+n)

= {h2 (m + n) +(d2 - h2)(m + n)} + mn(m+n)

= {h2 (m + n) +d2(m + n) -h2(m + n)} + mn(m+n)

= h2 (m + n) and - h2(m + n) cancels out

= {~~h2 (m + n)~~ +d2(m + n) -~~h2(m + n)~~} + mn(m+n)

$$= d^2(m + n) + mn(m + n)$$

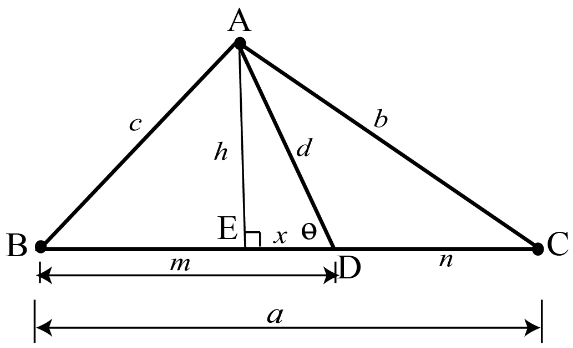
= (m + n)(h2 + n2) + (m + n)(d2 - h2)

= (m+n) (h2+ n2 + d2 - h2)

= (m+n) (n2 + d2)

$$b^2 m + c^2 n = d^2(m + n) + mn(m + n) \qquad (2.8)$$

$$man + dad = bmb + cnc \qquad (2.9)$$



|AD|=m |DC|=n |AE|=h |ED|=x

Figure 2: Using Pythagorean Theorem

Special Cases

Apollonius' Theorem

Apollonius' Theorem is named for Apollonius of Perga, an ancient Greek mathematician. This theorem is a special case of Stewart's Theorem, when the cevian is a median. In this situation, the segments *BD* and *DC* would both have a length of *m*. See Figure 3. Substituting these values into Stewart's Theorem, we would write

$$m^2(2m) + d^2(2m) = bmb + cmc \qquad (3.1)$$

Dividing both sides by m yields

$$2m^2 + 2d^2 = b^2 + c^2 \qquad (3.2)$$

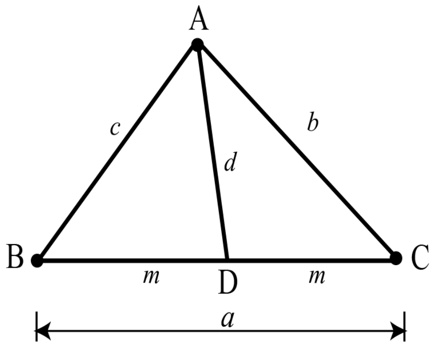


Figure 3: Apollonius' Theorem

Angle Bisector

Another special case would be if $\overline{AD}$ is an angle bisector. See Figure 4. By Stewart's Theorem, we have

$$man + dad = bmb + cnc$$

We also have, using the Angle Bisector Theorem,

$$\frac{b}{n} = \frac{c}{m} \qquad (3.3)$$

$$bm = cn \qquad (3.4)$$

We can then substitute, factor, and cancel.

$$man + dad = (cn)b + (bm)c \qquad (3.5)$$

$$a\left(mn + d^2\right) = bcn + bcm \qquad (3.6)$$

$$a\left(mn + d^2\right) = bc(m + n) \qquad (3.7)$$

Replace m + n by a.

$$a\left(mn + d^2\right) = bca \qquad (3.8)$$

Canceling the common factor a, it becomes

$$mn + d^2 = bc \qquad (3.9)$$

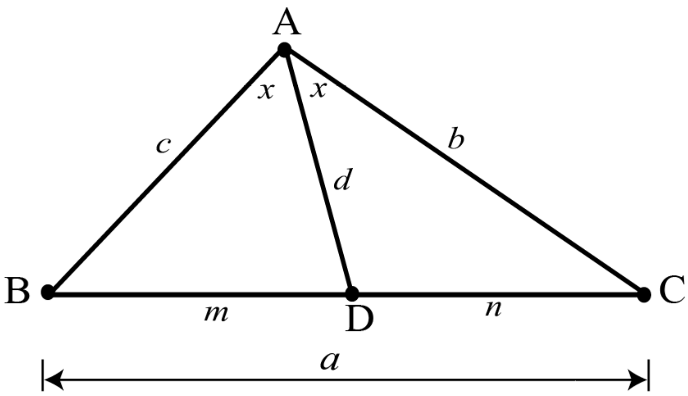


Figure 4: Angle Bisector

Altitude

If the cevian is an altitude, then we would be able to use the Pythagorean Theorem twice. We have the equations

$$c^2 = m^2 + d^2 \qquad (3.10)$$

$$b^2 = d^2 + n^2 \qquad (3.11)$$

$$man + d^2a = b^2m + c^2n \qquad (3.12)$$

We then substitute the first two equations into the third.

$$man + d^2a = \left(d^2 + n^2\right)m + \left(m^2 + d^2\right)n \qquad (3.13)$$

$$man + d^2a = d^2m + n^2m + m^2n + d^2n \qquad (3.14)$$

$$man + d^2a = d^2(m + n) + mn(m + n) \qquad (3.15)$$

Note that $m + n$ is merely a. This verifies Stewart's Theorem when the cevian is an altitude.

Isosceles Triangle

If the triangle is isosceles, with the cevian intersecting the base, then both sides could be labeled as length *b,* resulting in the equation

$$man + dad = bmb + bnb$$

This can be simplified to

$$man + dad = b^2(m + n) \qquad (3.16)$$

Replace m + n by a on the left hand side.

$$man + dad = b^2a \qquad (3.17)$$

Factor out *a* and cancel it.

$$a\,(mn + d^2) = b^2a \qquad (3.18)$$

$$mn + d^2 = b^2 \qquad (3.19)$$

Equilateral Triangle

An equilateral triangle is also a special case of an isosceles triangle. In this case, the base would have a length of *b* as well. Therefore the equation could be written as

$$m(b - m) + d^2 = b^2 \qquad (3.20)$$

Extension 1

Two Cevians From the Same Vertex

I explored the idea of having two cevians drawn from the same vertex and deriving a formula for this triangle. See Figure 5.

I applied Stewart's Theorem to ΔABE and ΔACB with $\overline{AD}$ and $\overline{AE}$ as cevians respectively. Note AB is now b instead of c, and AC is c instead of b. As a result, we have

For the $\Delta ABE$with $\overline{AD}$ as cevians

Since m=m+n, n=o, b=c, c=b, d=e, the Stewart's Theorem becomes

$$mn(m + n) + d^2(m + n) = b^2n + e^2m \qquad (4.1)$$

For the ΔACB with $\overline{AE}$ as cevians

Since m=m+n, n=o, b=c, c=b, n=o, the Stewart's Theorem becomes

$$(m + n)oa + e^2a = c^2(m + n) + b^2o \qquad (4.2)$$

In the first equation, we could solve for $m + n$:

$$m + n = \frac{b^2n+e^2m}{mn+d^2} \qquad (4.3)$$

Shifting the terms around in the second equation and factoring result in

$$e^2a - b^2o = (m + n)\left(c^2 - oa\right) \qquad (4.4)$$

Afterwards, we substitute $\frac{b^2n+e^2m}{mn+d^2}$ for $m + n$ into the second equation and cross multiply.

$$\left(e^2a - b^2o\right)\left(mn + d^2\right) = \left(b^2n + e^2m\right)\left(c^2 - oa\right) \qquad (4.5)$$

We then expand and factor, ending up with

$$e^2amn + e^2ad^2 - b^2omn - b^2od^2 = c^2b^2n - b^2noa + e^2mc^2 - e^2moa$$

$$e^2amn + e^2ad^2 + e^2moa - e^2mc^2 = b^2od^2 + b^2omn + c^2b^2n + b^2noa$$

$$e^2(man + dad + mao - cmc) = b^2(dod + mon + cnc - nao) \qquad (4.6)$$

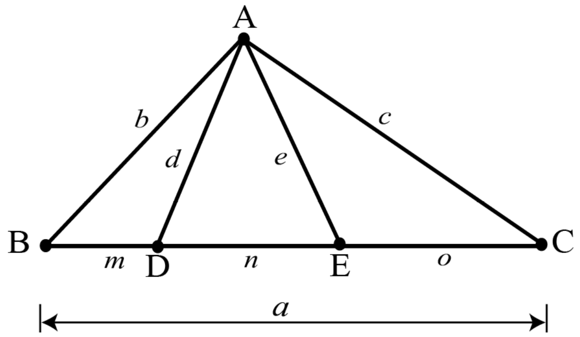


Figure 5: Two Cevians From the Same Vertex

Extension 2

Three Cevians From the Same Vertex

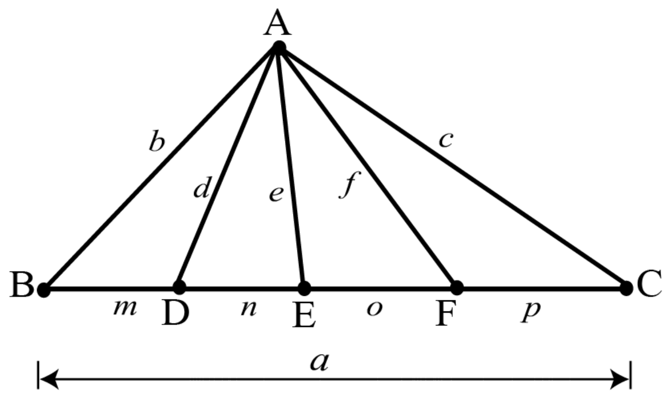


Figure 9: Three Cevians From the Same Vertex

1) Stewart's Theorem is used on ΔAEC
   a) $e^2p + c^2o = (o+p)(f+op)$

2) Solve for e^2
   a) $e^2 = (of + pf + o^2p + op^2 - c^2o)/p$

3) Substitute 2-(a) to the equation from two cevians from the same vertex
   a) $e^2(man+dad+mao-cmc) = b^2(dod+mon+cnc-nao)$
   → equation from two cevians
   b) $(of + pf + o^2p + op^2 - c^2o)\ (man + dad + mao - cmo)$
   $= pb^2(dod + mon + cnc - nao)$

4) Expand & Factor
   a) $[(o+p)(f+op)n - c^2o](man + dad + mao - cmc)$
   $\mathbf{= pb^2(dod + mon + cnc - nao)}$

Part B. Relationships Between Perimeter Formed by Centroids of Cevasix Triangles and the Perimeter of the Original Triangle

In this section, centroids, incenters, excenters, circumcenters of cevasix triangles were studied to find new relationships between these centers and perimeters of the polygon formed by centroids of cevasix triangles.

Let D, E and F be points on the side of triangle [BC], [CA] and [AB]. When cevians [AD], [BE] and [CF] all concur at the point P, let's call the six triangles BPD, CDP, CEP, ,AEP, AFP, BFP as “Cevasix Triangles”.

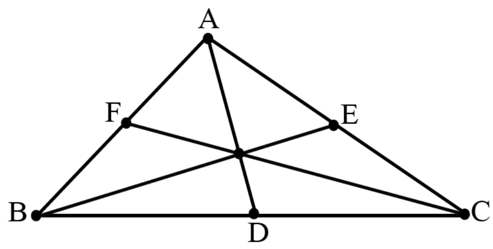


Figure 10: Three cevians from the same vertex - cevasix triangles

In the Figure below, the triangle ABC is orthic triangle DEF:

m(FDB} = m(EDE) = m(A)

m(AEF) = m(CED) = m(B)

m(AFE) = m(DFB)= m(C)

In orthic triangle DEF, h_a is the height of side the triangle and we can write:

Perimeter(DEF)=$(2h_a)\sin(A)$

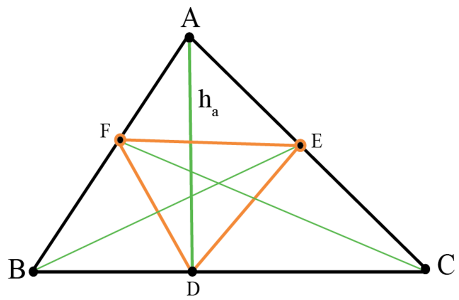


Figure 11: Orthic triangle

What is the relationship between the perimeters?

Let the centroids of cevasix triangles BDP, CDP, CEP, AEP, AFP, and BPF be respectively X1, X2, Y1, Y2, Z1, and Z2.

Also assume the semi perimeter of the hexagon X1, X2, Y1, Y2, Z1, Z2 formed by all the centroids is *s* and the semi perimeter of ABC is u. The altitude from vertex A is h_a.

The following inequality can be proved.

$U + (2h_a)\sin(A) \leq 3s$

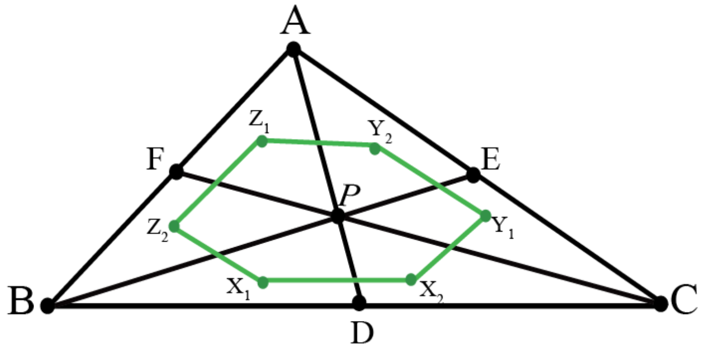


Figure 11: Cevasix triangle

First, assume X_1, X_2, Y_1, Y_2, Z_1, and Z_2 are the centroids of each cevasix triangle.

$X_1PX_2 \sim MPL$

$|X_1X_2| = ⅔\ ML = ⅔\ |BC|/2 = |BC|/3$

$|X_1X_2|+|X_1X_2|+|X_1X_2|$

$= ⅓(|AB|+|BC|+|CA|)$

$=2u/3$

And, from cevasix triangles CPE and CPD

$|PY_1|/PK| = |PX_2|/PL| = ⅔$

From the similarity,

$|Y_1X_2| // |KL|$ and $⅔|KL| = |Y_1X_2|$

On the triangle DEC, K and L are the midpoints of the sides.

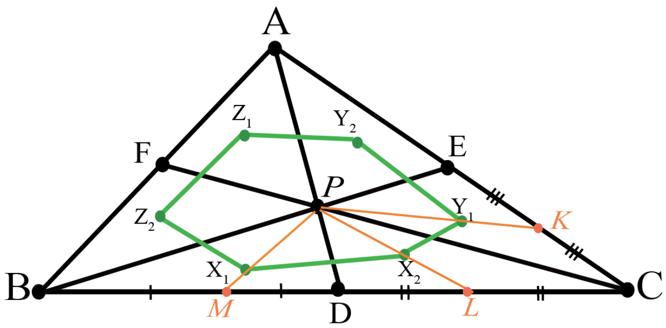


Figure 12: Cevasix triangle

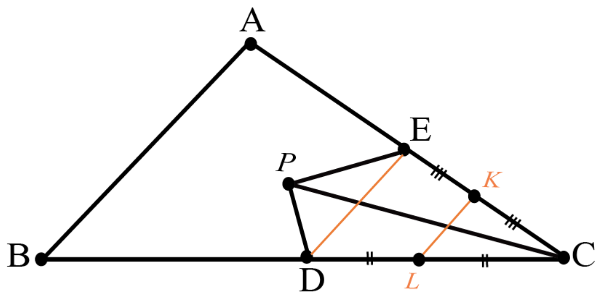


Figure 13: Cevasix triangle with [DE]//[KL]

From the similarity

[DE]//[KL] and

$\frac{1}{2}$ |DE| = |KL|

Thus,

$|Y_1X_2| = 2/3|KL| = (\frac{2}{3})\ |DE|/2 = |DE|/3$

And,

$|Y_1X_2|+|Z_1Y_2|+|X_1Z_2|$

$= \frac{1}{3}(|DE|+|EF|+|FD|)$

=Perimeter(DEF)/3

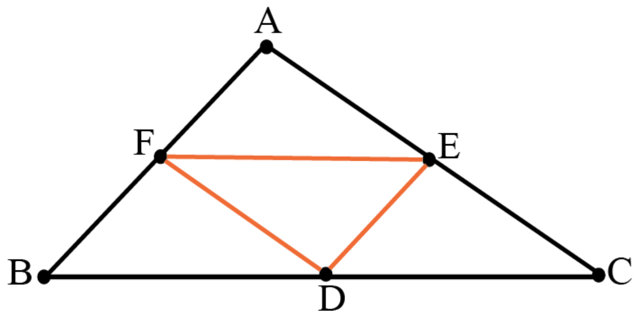


Figure 14: Cevasix triangle with triangle DEF

Thus, the perimeter surrounded by X_1, X_2, Y_1, Y_2, Z_1 and Z_2 can be found in terms of u and t.

Perimeter($X_1X_2Y_1Y_2Z_1Z_2$) = 2s

= 2u/3 + Perimeter(DEF)/3

Perimeter(DEF) = 2t

2s = 2u/3 + 2t/3

Finally,

3s = u+t

Or, if DEF is the orthic triangle of ABC,

Perimeter(DEF) = 2t =($2h_a$) sin(A)

3s = u+ $h_a$sin(A)

It would be interesting to see the variations or changes of the behavior of this equation as the three variables change, and it seems like a daunting task to draw the graph.

Discussion and Conclusion

Cevians and their following formulas in a cevasix triangle were studied during this project. All formulas/equations were conducted mainly according to Stewart's Theorem with its equation: man + dad = bmb + cnc. At first, the project discussed how a certain law/theorem, which we are already familiar with, can be utilized to conclude with the equation of Stewart’s Theorem; I specifically referred to the law of cosines and the Pythagorean theorem.

Furthermore, I realized that the equation of Stewart's Theorem could be modified depending on different conditions. This allowed me to conclude that various special cases, two cevians, and two intersecting cevians can contribute to the alternate equation. Special cases, like Apollonius' Theorem, angle bisector, altitude, isosceles triangle, and equilateral triangle, were useful because they could make the equation the simplest form. While two cevians and two intersecting cevians do not produce a simple equation, they explain how Stewart's theorem is versatile and can fit into various and interesting cases.

In conclusion, I also discovered that deriving formulas with n-cevians from the same vertex are possible by using the derived equation for (n-1). One example is deriving an equation with three cevians from the equation for two cevians from the same vertex.

Lastly, a relationship between the perimeter formed by centroids of cevasix triangles and the perimeter of the original triangle was found. Centroids, incenters, excenters, circumcenters of cevasix triangles were studied to find new relationships between these centers and perimeters of the polygon formed by centroids of cevasix triangles.

As a future consideration, using the Trigonometric Form of Ceva's Theorem, I will perform further study about the relation: The relation between the area of the hexagon formed by connecting six centroids and the one of a starting triangle.

Through this project, I learned how the use of Stewart's Theorem can be formalized by various theorems and laws and further extend itself by applying it to other polygons in relation to areas of triangles in the polygon.

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