STRICT INEQUALITIES FOR THE n-CROSSING NUMBER

NICHOLAS D. HAGEDORN

Abstract. In 2013, Adams introduced the n-crossing number of a knot K, denoted by $c_n(K)$. Inequalities between the 2-, 3-, 4-, and 5-crossing numbers have been previously established. We prove $c_9(K) \leq c_3(K) - 2$ for all knots K that are not the trivial, trefoil, or figure-eight knot. We show this inequality is optimal and obtain previously unknown values for $c_9(K)$.

1. Introduction

In knot theory, a standard crossing in a knot diagram is when one strand passes over another. In [1], Adams introduced an n-crossing projection, which is a knot diagram where n strands intersect at each crossing, with each strand bisecting the crossing. Every knot has an n-crossing projection for all $n \geq 2$. One can then define $c_n(K)$ as the smallest number of crossings in an n-crossing projection of K. If K is a non-trivial knot, $c_n(K) \geq 1$.

Inequalities between these n-crossing numbers have been studied: The construction in Figure 1 shows that every crossing of n strands can be turned into one with $n+2$ strands. Thus, an n-crossing can be made into an $(n+2)$-crossing and $c_{n+2}(K) \leq c_n(K)$. Interestingly, $c_3(K) \leq c_2(K) - 1$ for 2-braid knots K and $c_3(K) \leq c_2(K) - 2$ for all other non-trivial knots [1]. In fact, Jabłonowski proved that $c_3(K) = c_2(K) - 1$ for all 2-braid knots [8]. For a non-trivial knot K, $c_4(K) \leq c_2(K) - 1$ [2] and $c_5(K) \leq c_3(K) - 1$ [3].

Until now, specific n-crossing numbers for $n > 5$ have not been studied. In Section 2, we deduce upper bounds on $c_9(K)$ for knots with certain 5-crossing projections and obtain new values for $c_9(K)$. In Section 3, we prove that $c_9(K) \leq c_3(K) - 2$ for all knots K that are not the trivial, trefoil, or figure-eight knot. We also show this inequality is optimal.

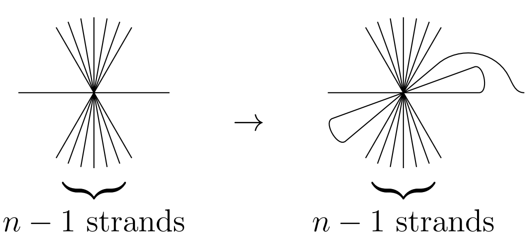

Figure 1. Turning an n-crossing into an $(n+2)$-crossing

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2. Classification of 5-crossing Knots

Adams introduced the notion of a crossing covering circle in [1]. For some knot projection P, a crossing covering circle is a topological circle in the projection plane that only intersects P at crossings, and when it intersects a crossing, it passes straight through the crossing. We introduce two definitions that generalize this concept.

Definition 2.1 (Crossing Segment)**.** *Take an n-crossing knot projection P. Define a crossing segment to be a topological line segment in the projection plane that only intersects P at crossings. When the crossing segment does intersect P, it must pass directly through the crossing, with n strands of the crossing on either side of P. Lastly, a crossing segment cannot intersect a crossing more than once.*

Definition 2.2 (Crossing Connected)**.** *For a given n-crossing knot projection P, we say two faces in P are crossing connected via α if they have common vertex α (a crossing) and can be connected by a crossing segment that does not intersect α. Faces are adjacent crossing connected (ACC) via α if they are crossing connected via α and both border a strand that connects α to another crossing (the top of Figure 2 shows two ACC faces). Faces are opposite crossing connected (OCC) via α if they are crossing connected via α and are on opposite sides of α (the top of Figure 3 shows two OCC faces).*

Note that the crossing segment connecting two opposite crossing connected faces is equivalent to a crossing covering circle by extending the crossing segment through the crossing. Next, we will need the following known lemma to gain insights into ACC and OCC faces.

Lemma 2.3. *If all vertices of a planar graph G have even degree, then the dual graph $\hat{G}$ is 2-colorable.*

Proof. Recall that a graph is 2-colorable precisely when it has no odd cycles. Now suppose $\hat{G}$ has an odd cycle. Then $\hat{G}$ contains a simple closed odd cycle. As the vertices of G have even degree, all the faces of $\hat{G}$ have an even number of edges. Then the sum total of the number of edges of all faces inside a simple closed odd cycle must be even. As the edges on the interior of the cycle are counted twice, there must be an even number of edges in the cycle. But then the cycle is even, contradicting the assumption that it is odd. Thus $\hat{G}$ does not have an odd cycle and $\hat{G}$ is 2-colorable. □

Lemma 2.4. *Let n be odd. In an n-crossing projection of a knot, a crossing segment connecting two ACC or OCC faces passes through an odd number of crossings.*

Proof. Every crossing in an n-crossing knot projection has degree $2n$. So, Lemma 2.3 gives that there exists a checkerboard coloring for the faces of any n-crossing knot projection. It is easy to see that opposite and adjacent sides are different colors for odd n. And since, for odd n, face color alternates when the crossing segment passes through a crossing, a crossing segment connecting two ACC or OCC faces passes through an odd number of crossings. ☐

Theorem 2.5. *Given an n-crossing projection of knot K with odd n and two adjacent or opposite crossing connected faces, there exists a $(2n-1)$-crossing diagram of K with one fewer crossing.*

Proof. Regardless of whether there is an ACC or OCC, we know the crossing segment intersects an odd number of crossings due to Lemma 2.4. Then, if two faces are ACC via a crossing α, we can pull the $n-1$ strands of α that don't border both ACC faces around the crossing segment. This move is shown in Figure 2 for the $n = 5$ case. Similarly, if two faces are OCC via a crossing α, we can pull the $n-2$ strands of α that don't border both OCC faces and one of α's two strands that do border α around the crossing segment. This move is shown in Figure 3 for the $n = 5$ case. Both moves eliminate a crossing while turning the n-crossings along the crossing segment into $(2n-1)$-crossings. We can then perform the move shown in Figure 1 a total of $\frac{n-1}{2}$ times on all remaining crossings to convert them from n-crossings to $(2n-1)$-crossings. This construction provides a $(2n-1)$-projection of K with one fewer crossing than the original n-crossing projection. ☐

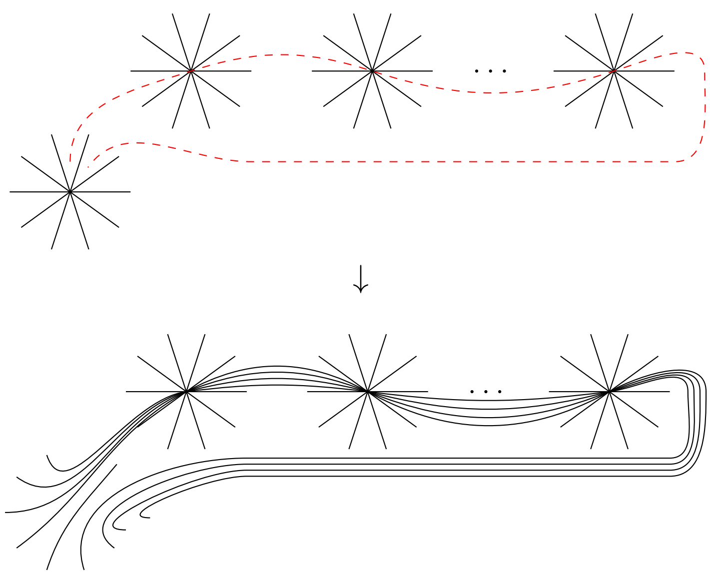

FIGURE 2. An n-crossing projection with two ACC faces can be transformed into an $(n+2)$-crossing projection with one fewer crossing

Limiting the n-crossing projection of K in Theorem 2.5 to an n-crossing projection with $c_n(K)$ crossings, we achieve the following result:

Corollary 2.6. *For odd n and knot K with $c_n(K) > 1$, if there exists a minimal n-crossing projection with two adjacent or opposite crossing connected faces, then $c_{2n-1}(K) < c_n(K)$.*

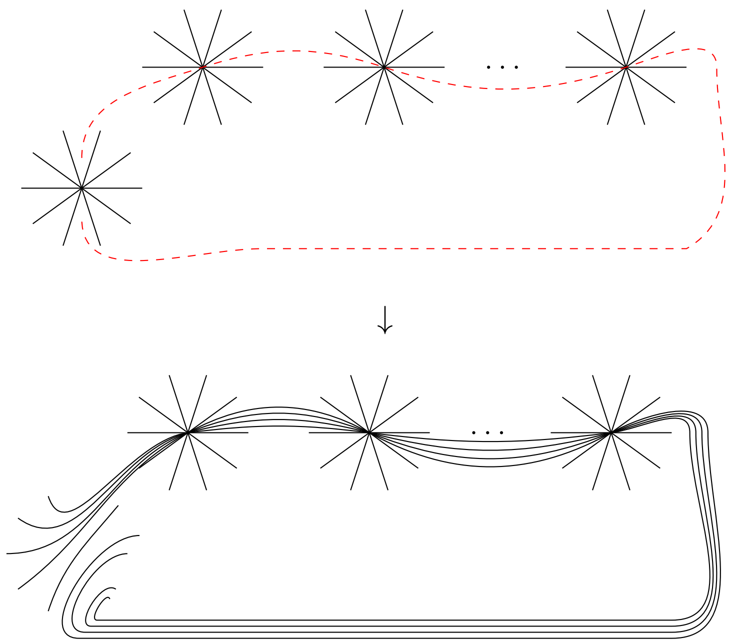

FIGURE 3. An n-crossing projection with two OCC faces can be transformed into an $(n+2)$-crossing projection with one fewer crossing

Though the above results hold for n-crossing projections for any odd n, the next part of Section 2 will focus on results specific to the 5- and 9-crossing numbers. In addition, since $c_{n+2}(K) \leq c_n(K)$ for all n, we know that $c_9(K) \leq c_5(K)$ for any knot K. So, we are primarily interested in determining if there exist any knots K such that $c_9(K) = c_5(K)$.

Definition 2.7 (Adjoined Bigon)**.** *Define an adjoined bigon to be a polygon with one vertex and two edges, as seen in Figure 4.*

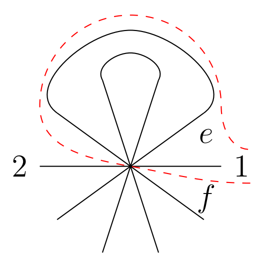

FIGURE 4. An adjoined bigon as part of a 5-crossing diagram

Corollary 2.8. *If a minimal 5-crossing projection of K for knot K with $c_5(K) > 1$ contains an adjoined bigon, then $c_9(K) < c_5(K)$.*

Proof. We will use the labeled strands of the adjoined bigon seen in Figure 4. It is easy to see that if both strand 1 and strand 2 connect back to the original crossing, forming a monogon or adjoined bigon, K will either be a link with more than one component or have $c_5(K) = 1$. So, strand 1 or 2 must connect

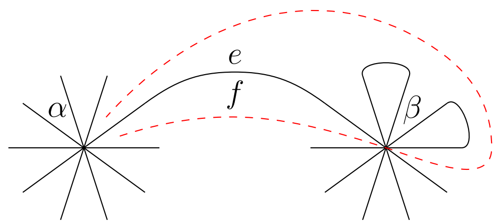

FIGURE 5. Crossing with 2 consecutive monogons

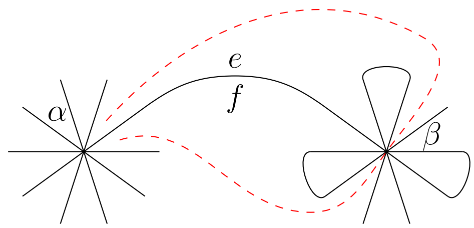

FIGURE 6. Crossing with three non-consecutive monogons

to some other crossing—call it β. Without a loss of generality, assume that strand 1 connects to crossing β. This means that faces e and f are ACC via crossing β, and $c_9(K) < c_5(K)$. □

Corollary 2.9. *If a minimal 5-crossing projection of K for knot K with $c_5(K) > 1$ contains a crossing with three or more monogons, then $c_9(K) < c_5(K)$.*

Proof. A crossing of K can't have five monogons as then $c_5(K) = 1$. If a crossing of K has four monogons, two of the monogons must be consecutive. This gives the crossing seen in Figure 5. Faces e and f are ACC via α, and we are done. The only crossing with at least three monogons and no two consecutive monogons is seen in Figure 6. Again, faces e and f are ACC via α, which means $c_9(K) < c_5(K)$. □

Definition 2.10 (Almost Opposite)**.** *Two monogons are almost opposite if they share the same vertex and if there are exactly two strands in between them.*

These three corollaries give the following theorem:

Theorem 2.11. *Let K be a knot with $c_5(K) > 1$. Then $c_5(K) = c_9(K)$ can only hold if the following three conditions hold in every minimal 5-crossing projection of K: (1) every crossing has either no monogons, one monogon, or two almost opposite monogons; (2) no crossing has an adjoined bigon; and (3) no two faces are ACC or OCC.*

Proof. Parts (2) and (3) have already been proven in Corollary 2.8 and Corollary 2.6, respectively. Corollary 2.9 gives that $c_9(K) = c_5(K)$ only if all crossings have at most two monogons. Consider the case where a

Knot	$c_5(K)$	$c_9(K)$
8_5	2	1
8_{17}	2	1
8_{18}	2	1
$4_1\#4_1$	2	1
9_{47}	2	1
9_{49}	2	1

Table 1. Knots with $c_5(K) = 2$ and previously unknown 9-crossing number

crossing has two monogons. The monogons can't be opposite as K would be a link with more than one component. If the monogons are consecutive or separated by one strand, this gives crossings similar to the ones in Figure 5 and Figure 6. But then face e and f are ACC via α, and $c_9(K) < c_5(K)$. Then every crossing in the minimal 5-crossing diagram of K must be a crossing with no monogons, a crossing with one monogon, or a crossing with two almost opposite monogons. So the claim holds. □

Corollary 2.12. *If knot K has $c_5(K) = 2$, then $c_9(K) = 1$.*

Proof. Consider a 5-crossing diagram of K with two crossings. By Theorem 2.11, each crossing can contain at most two monogons. With only two crossings, both crossings must have the same number of monogons. Disregarding crossing information, there are fifteen total knot diagrams with two crossings and where each crossing has either no monogons, one monogon, or two almost opposite monogons. In each of the fifteen cases, one can check that there exists two ACC or OCC faces. Then by Theorem 2.11, $c_9(K) = 1$. □

Corollary 2.12 allows us to obtain previously unknown values of $c_9(K)$ for a few knots. Multi-crossing tabulation in [4] obtained the fifth crossing number but not the ninth crossing number of the knots in Table 2.

3. A 9-crossing Number Inequality

In this section, we will prove:

Theorem 3.1. *Let K be a knot that is not the trivial, trefoil, or figure-eight knot. Then*

$$c_9(K) \leq c_3(K) - 2.$$

The reason we have the restriction on K is because the trivial, trefoil, and figure-eight knot are exactly the knots such that $c_3(K) \leq 2$ [7]. As $c_9(K) \geq 1$, the inequality requires that we exclude these knots. We then prove the inequality for all other knots.

We will establish Theorem 3.1 by proving three propositions. We consider the case where a minimal triple-crossing diagram of K contains two or more monogons, contains one monogon, and contains zero monogons.

Proposition 3.2, Proposition 3.4, and Proposition 3.9 respectively prove the inequality in each case.

We start with the case where a minimal triple-crossing diagram of K contains two or more monogons.

Proposition 3.2. *Let K be a knot that is not the trivial, trefoil, or figure-eight knot. If a minimal 3-crossing diagram of K contains at least two monogons, then $c_9(K) \leq c_3(K) - 2$.*

Proof. For the minimal 3-crossing diagram of K with two monogons, take a crossing α that contains a monogon. The crossing α cannot be the crossing seen in Figure 7 because then it would be a link with more than one component. The crossing also can't be the crossing seen in Figure 8 as no matter how the crossing information is filled in, the crossing could be removed by the 3-crossing 1-move described in [3, p. 3]. The remaining case is then the crossing seen in Figure 9. Since α has only one monogon, there exists at least one other crossing with a monogon. Let β be such a crossing.

Call the crossings that connect to α by the 1 and 2 strands its adjacent crossings. Crossing β has similar adjacent crossings. All the adjacent crossings of α and β can't be α and β as that would make K a link with more than one component, as seen in Figure 10. So, without a loss of generality, let α be the crossing that contains at least one adjacent crossing that is not the other crossing. So, take an adjacent crossing to α that is not β. Call it

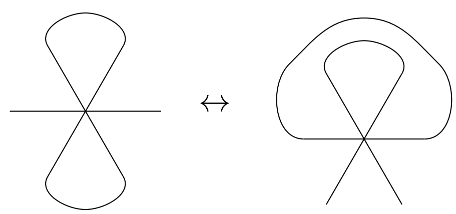

FIGURE 7. Crossings that are part of a link with more than one component

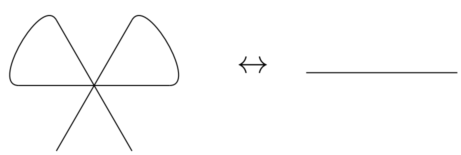

FIGURE 8. Crossing that can be removed with a 3-crossing 1-move

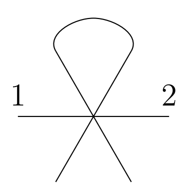

FIGURE 9. Diagram of α

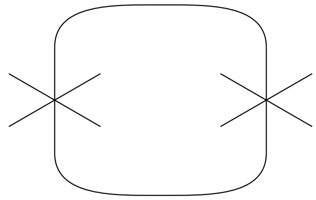

FIGURE 10. If β and α's 1 and 2 stands are joined, K is a link with more than one component

γ. There now exists an ACC via γ. We can wrap strands of γ around α to eliminate the former crossing while converting the latter to a 5-crossing, as described in [3, p. 16].

Let us also transform all other 3-crossings to 5-crossings using the move shown in Figure 1. This adds two monogons to all these crossings, leaving β with at least three monogons. Now this is one of the crossings detailed in Theorem 2.11, meaning there exists a 9-crossing projection of K with one fewer crossing than the 5-crossing projection. The 9-crossing projection has one fewer crossing than the 5-crossing projection, which in turn has one fewer crossing than the minimal 3-crossing projection. Hence $c_9(K) \leq c_3(K) - 2$. □

Lemma 3.3. *Let K be a knot that is not the trivial, trefoil, or figure-eight knot whose minimal 3-crossing diagram:*

(1) Contains a crossing α with a monogon; and

(2) Contains two faces that are crossing connected via a crossing different from α through a crossing segment that does not intersect α.

Then $c_9(K) \leq c_3(K) - 2$.

Proof. By the methods used in Theorem 2.5 and [3, p. 17], we can transform the 3-crossings around the crossing connection into 5-crossings, eliminating one crossing in the process. We will now apply the transformation seen in Figure 1 to all the remaining 3-crossings, which includes α. This results in $c_3(K) - 1$ crossings, all of which are 5-crossings. The second move adds two monogons to a crossing, meaning α contains three monogons. As explained in the proof of Corollary 2.9, this 5-crossing projection contains two ACC faces. By Theorem 2.5, there exists a 9-crossing projection with one fewer crossing. Thus,

$$c_9(K) \leq (c_3(K) - 1) - 1 = c_3(K) - 2.$$

□

Proposition 3.4. *Let K be a knot that is not the trivial, trefoil, or figure-eight knot. If a minimal 3-crossing diagram of K contains exactly one monogon, then $c_9(K) \leq c_3(K) - 2$.*

Proof. Consider a 3-crossing projection of K with $c_3(K)$ crossings and one monogon. Let α be the crossing containing the monogon, and label its adjacent faces as seen in Figure 11. We will separately consider the cases

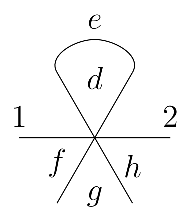

FIGURE 11. Labeling the faces adjacent to α

when face g is and isn't a bigon. In each case, we will show that there exist two faces satisfying conditions (1) and (2) of Lemma 3.3. Consequentially, $c_9(K) \leq c_3(K) - 2$.

Case 1: *Face g is a bigon.* Refer to the labels shown in Figure 12. If both f and h are bigons, then the outer edges of f and h would form a link component, and K wouldn't be a knot. So one of f and h is not a bigon. By symmetry, we can assume that face f is not a bigon and, consequentially, that f has an edge connecting α to a crossing γ that is different from β. By Lemma 3.6, there is a loop q on the 3-crossing projection that (i) connects crossings by passing through faces, only intersecting the strands of the knot projection at crossings; (ii) bisects at least $n-1$ of the n crossings it intersects; (iii) doesn't intersect α; and (iv) doesn't intersect any crossing more than once.

Case 2: *Face g isn't a bigon.* Let m be the number of edges of face e. We first show that we need only consider the situation when $m > 3$. The case when $m = 1$ cannot occur as d is the only monogon in the projection. The case when $m = 2$ also cannot occur, for if face e has two sides, then K is a link with more than one component. Now suppose that $m = 3$. Then, since face e has three sides, the diagram must look like the left side of Figure 13. Using Lemma 8 of [9], the move in Figure 13 makes face g a bigon. This situation is then covered by Case 1. We can then assume that face e has $m > 3$ edges. The projection must then be as in Figure 14. Using the figure's labels, note that crossings ε and β might be the same and crossings δ and γ might be the same. By Lemma 3.8, there is a loop q on the 3-crossing projection that (i) connects crossings by passing through faces, only intersecting the strands of the knot projection at crossings; (ii) bisects at least $n-1$ of the n crossings it intersects; (iii) doesn't intersect α; and (iv) doesn't intersect any crossing more than once.

In each case, we have constructed a loop q that satisfies the above four criteria. We now modify q to find the path needed for Lemma 3.3. Loop q never intersects a crossing more than once. Then if q self-intersects with itself, this self-intersection must occur at a point z on a face. Let loop l be the part of q from the first time it intersects z to the second time it intersects z. If l doesn't intersect a crossing, we modify q by replacing l with the point z. We do this with all loops l in q that don't intersect a crossing. Thus, any loop coming from a self-intersection of q must intersect a crossing.

From graph theory, the loop q contains a simple cycle l. By the modification of q above, l must intersect at least one crossing. Since l is a subloop of q and q satisfies property (ii) above, l also satisfies (ii) and there exists at most one crossing that l intersects but doesn't bisect. If this crossing exists, call it δ. If it doesn't exist, let δ be an arbitrary crossing that l intersects.

Then removing δ from the loop l, we obtain a crossing segment as l never intersects a crossing more than once, and l now bisects all crossings it intersects. This crossing segment connects two faces that are crossing connected via δ and doesn't intersect α (since q satisfies property (iii)). Then the crossing segment satisfies conditions (1) and (2) of Lemma 3.3. Hence $c_9(K) \leq c_3(K) - 2$.

□

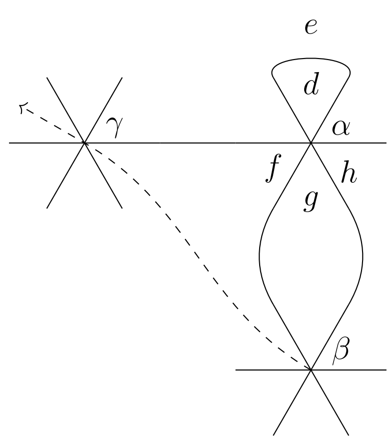

FIGURE 12. Face g is a bigon

Lemma 3.5. *Let $\mathcal{C}$ be a 3-crossing projection of knot K with one monogon. Assume that the monogon borders the crossing α and that the faces near α are labeled as in Figure 12, with face g a bigon and face f having more than two vertices. Let p be a path on $\mathcal{C}$ that starts at the crossing β and initially*

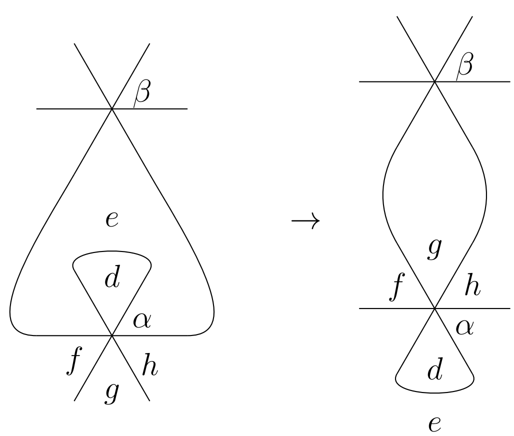

FIGURE 13. Face e is a triangle

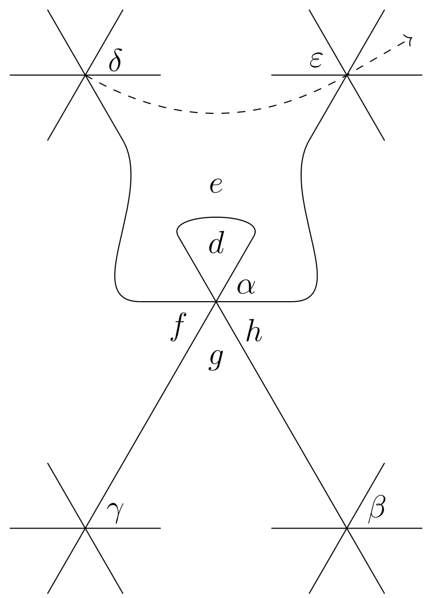

FIGURE 14. Face g is not a bigon, and face e has more than three edges

travels through the face f to crossing γ, as illustrated in Figure 12. Assume that p satisfies these rules:

1. *The path p travels between crossings by passing through faces and will only intersect strands of the knot projection at crossings.*
2. *When p intersects a crossing, it travels to the exact opposite side of the crossing, bisecting the crossing.*

Assume further that p ends by intersecting a crossing $\zeta \neq \alpha$ into a face z and p has never previously intersected ζ. Then the face z has a crossing $\zeta' \neq \zeta, \alpha$.

Proof. Assume that the face z does not border any crossings that are distinct from ζ and α.

If z just borders ζ, then z is a monogon or an adjoined bigon. But z can't be an adjoined bigon as then K isn't a knot, as shown in Figure 7. So z is a monogon. As face d is the only monogon, then $z = d$. Now, the path p can only reach face d by intersecting the crossing α. Hence $\zeta = \alpha$. As this contradicts the hypothesis on ζ, face z cannot just border ζ.

Now suppose that z only borders the crossings ζ and α. Then face z must border α and have only two vertices. Since face f borders at least three vertices, from Figure 12, face z must be e, g, or h. We will obtain a contradiction in each case.

Suppose $z = g$. Since $\zeta \neq \alpha$ by hypothesis, p can only reach face g by intersecting $\zeta = \beta$. But as p started at β, this would be the second intersection of β, which contradicts the assumption that p has never previously intersected ζ.

Suppose $z = e$. Since z only borders ζ and α, e only borders γ and α. Because $\zeta \neq \alpha$, path p must have intersected $\zeta = \gamma$ to reach face e. Because

d is the only monogon, γ does not have a monogon. So γ borders six unique faces and the face opposite f through γ is not e. This shows that p reaches e after intersecting the crossing $\zeta = \gamma$ for a second time, which contradicts the hypothesis on ζ.

Suppose $z = h$. Since z only borders ζ and α, and h borders both β and α, then $\zeta = \beta$. But as p began at β and p never previously intersected ζ, $\zeta \neq \beta$, giving a contradiction.

As we found a contradiction in all cases, the assumption is false and the face z has a crossing ζ' that is distinct from α, ζ. □

Lemma 3.6. *Let $\mathcal{C}$ be a 3-crossing projection of knot K with one monogon. Assume that the monogon borders the crossing α and that the faces near α are labeled as in Figure 12, with face g a bigon and face f having more than two vertices. Then there is loop q on $\mathcal{C}$ such that:*

1. *Loop q connects crossings by passing through faces, only intersecting the strands of the knot projection at crossings.*
2. *Loop q bisects at least $n-1$ of the n crossings it intersects.*
3. *Loop q doesn't intersect α.*
4. *Loop q doesn't intersect any crossing more than once.*

Proof. We start with the path p that begins at β, traverses the face f, and travels into another face z by bisecting crossing γ. By Lemma 3.5, the face z has a crossing $\zeta' \neq \alpha, \gamma$. We extend p by traversing z and traveling to the opposite side of ζ', bisecting the crossing. If ζ' hasn't been previously intersected, we can continue this process, by applying Lemma 3.5 to extend p. As $\mathcal{C}$ has only finitely many crossings, eventually the path p intersects a crossing ζ' that it previously intersected. We then take the loop q to be the part of p from the first time it intersects ζ' to the second time it intersects ζ'. By construction, loop q only intersects strands of the knot projection at crossings, bisects every crossing it intersects that isn't ζ', doesn't intersect α as each crossing ζ' is distinct from α, and intersects each crossing at most once. □

Lemma 3.7. *Let $\mathcal{C}$ be a 3-crossing projection of knot K with one monogon. Assume that the monogon borders the crossing α and that the faces near α are labeled as in Figure 14, with face g having more than two vertices, and face e having more than three edges. Let p be a path on $\mathcal{C}$ that starts at the crossing δ and initially travels through the face e to crossing ε, as illustrated in Figure 14. Assume that p satisfies these rules:*

1. *The path p travels between crossings by passing through faces and will only intersect strands of the knot projection at crossings.*
2. *When p intersects a crossing, it travels to the exact opposite side of the crossing, bisecting the crossing.*

Assume further that p ends by intersecting a crossing $\zeta \neq \alpha$ into a face z and p has never previously intersected ζ. Then the face z has a crossing $\zeta' \neq \zeta, \alpha$.

Proof. Assume that the face z does not border any crossings that are distinct from ζ and α. The same proof as in Lemma 3.5 shows that z cannot just border ζ. Hence, z must only border the crossings ζ and α. Then face z must border α and have only two vertices. As illustrated in Figure 14, since face g and e both border at least three vertices, face z must be f or h. We will obtain a contradiction in each case.

Suppose $z = f$. Since z only borders ζ and α and f borders both δ and α, then $\zeta = \delta = \gamma$ (despite the drawing in Figure 14, δ and γ aren't necessarily distinct). But as p began at δ, this implies that $\zeta = \delta$ is a vertex that was previously crossed, contradicting the hypothesis on ζ.

Now suppose $z = h$. Since z only borders ζ and α, then $\zeta = \beta = \varepsilon$ (again, despite the figure, ε and β need not be distinct). But p previously intersected ε, which contradicts the assumption on $\zeta = \varepsilon$.

As we found a contradiction in all cases, the assumption is false and the face z has a crossing ζ' that is distinct from α, ζ. □

Lemma 3.8. *Let $\mathcal{C}$ be a 3-crossing projection of knot K with one monogon. Assume that the monogon borders the crossing α and that the faces near α are labeled as in Figure 14, with face g having more than two vertices, and face e having more than three edges. Then there is loop q on $\mathcal{C}$ such that:*

1. *Loop q connects crossings by passing through faces, only intersecting the strands of the knot projection at crossings.*
2. *Loop q bisects at least $n - 1$ of the n crossings it intersects.*
3. *Loop q doesn't intersect α.*
4. *Loop q doesn't intersect any crossing more than once.*

Proof. The proof is exactly the same as that for Lemma 3.6, except that we start with the path p that begins at δ, traverses the face e, and travels into another face z by bisecting crossing ε. We also use Lemma 3.7 instead of Lemma 3.5. □

Remark. Using the 3-crossing knot tabulation done in [7], we find that Proposition 3.2 and Proposition 3.4 account for all prime knots K with $c_3(K) = 3$. The only prime knots K with $c_3(K) = 4$ that Proposition 3.2 and Proposition 3.4 do not account for are the 10_{140}, $11n_{139}$, and $12n_{462}$ knots. Proposition 3.2 and Proposition 3.4 account for all prime knots K with $c_3(K) = 5$.

Proposition 3.9. *Let K be a knot that is not the trivial, trefoil, or figure-eight knot. If a minimal 3-crossing diagram of K contains zero monogons, then $c_9(K) \leq c_3(K) - 2$.*

Proof. Consider a 3-crossing diagram of K with $c_3(K)$ crossings. Let f_i represent the number of faces with i edges in a 3-crossing diagram. This includes the outer region. Adams, Hoste, & Palmer prove in [3] that $2f_1 + f_2 = 6 + f_4 + 2f_5 + 3f_6 + 4f_7 + ...$ using the Euler Characteristic. Since there are no monogons, we have

$$f_2 = 6 + f_4 + 2f_5 + 3f_6 + 4f_7 + \dots . \tag{1}$$

We will use proof by contradiction by assuming $c_9(K) > c_3(K) - 2$. We will then derive a contradiction with (1), proving the claim.

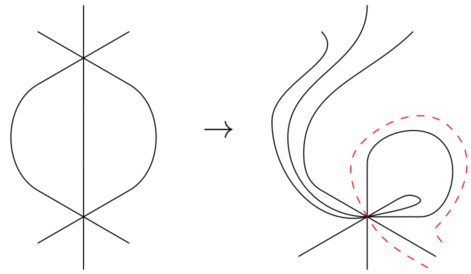

FIGURE 15. Two adjacent bigons reduce to a 5 crossing with an adjoined bigon

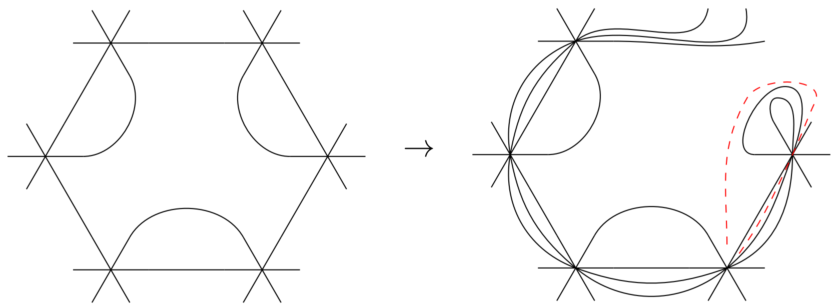

FIGURE 16. An n-gon with even n

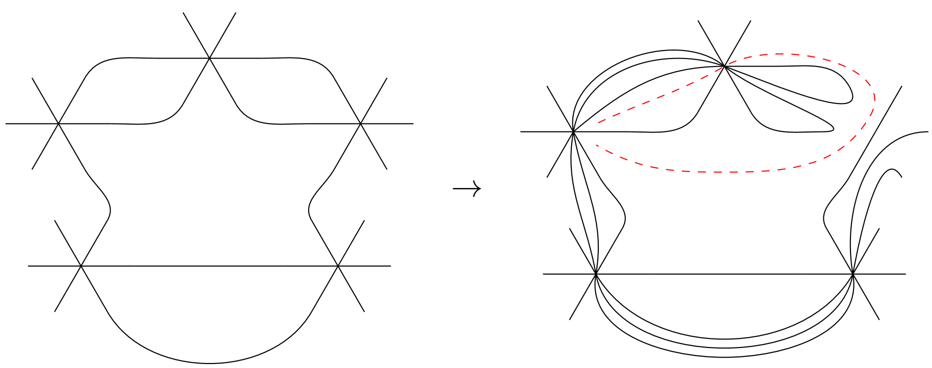

FIGURE 17. An n-gon with odd n

Note that two bigons can't be adjacent; otherwise, we can do the manipulations seen in Figure 15, meaning $c_9(K) \leq c_3(K) - 2$. So, we can count the total number of bigons by counting the number of bigons that can border each n-gon.

An n-gon must have fewer than $n-1$ adjacent bigons. If not, then at most one of the edges isn't adjacent to a bigon. But this means $c_9(K) \leq c_3(K) - 2$: If n is even, $c_9(K) \leq c_3(K) - 2$ by the construction seen in Figure 16. And if n is odd, $c_9(K) \leq c_3(K) - 2$ by the construction seen in Figure 17. So every n-gon borders at most $n - 2$ bigons.

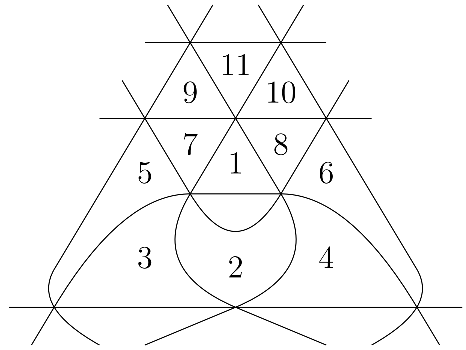

FIGURE 18. The maximum number of surrounding triangles

In counting bigons this way, we have counted them all twice. So, we obtain the inequality:

$$f_2 \leq \frac{1}{2}f_3 + f_4 + \frac{3}{2}f_5 + 2f_6 + \dots$$

To improve this bound and thus reach a contradiction with (1), look towards each triangle in the knot projection.

If two triangles share no vertices and are each adjacent to their own bigon, a move is described in [3, p. 17] that allows us to convert the triple-crossings of each triangle into 5-crossings. In the process, one crossing is eliminated from each triangle. Then $c_5(K) \leq c_3(K) - 2$ and $c_9(K) \leq c_3(K) - 2$.

Now, pick a triangle with an adjacent bigon—if no triangles have an adjacent bigon,

$$f_2 \leq f_4 + \frac{3}{2}f_5 + 2f_6 + \dots$$

and we achieve a contradiction with (1) regardless. There are at most ten other triangles that can share a crossing with our first triangle (see Figure 18), for a total of eleven triangles. Each of these triangles can have at most one adjacent bigon. Our new bound becomes

$$f_2 \leq \frac{11 + 2f_4 + 3f_5 + 4f_6 + \dots}{2}.$$

So,

$$f_2 \leq \frac{11}{2} + f_4 + \frac{3}{2}f_5 + 2f_6 + \dots. \tag{2}$$

This contradicts (1), meaning the claim must hold. □

We now prove Theorem 3.1.

Proof. Consider a minimal 3-crossing projection of a knot K that is not the trivial, trefoil, or figure-eight knot. Let $m \geq 0$ be the number of monogons in the projection. If $m \geq 2$, we are done by Proposition 3.2. If $m = 1$, we are done by Proposition 3.4. Lastly, if $m = 0$, we are done by Proposition 3.9. Since the inequality holds for all possible m, $c_9(K) \leq c_3(K) - 2$ for all knots K that are not the trivial, trefoil, or figure-eight knot. □

This inequality is optimal. [7] shows that $c_3(5_1) = 3$ and $c_3(6_2) = 3$. Since $c_9(5_1) = 1$ and $c_9(6_2) = 1$, these knots realize the upper bound.

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Princeton High School, Princeton New Jersey, 08540
Email address: nhagedorn@princetonk12.org