Strict Inequalities for the n-crossing Number

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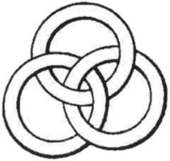
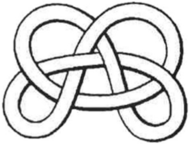
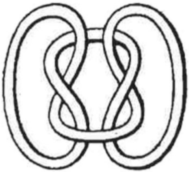






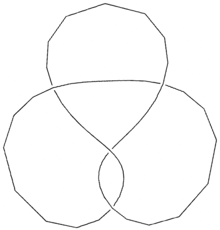
What is knot theory?

- The study of mathematical knots and how to distinguish one knot from another.
- Has applications in post-quantum encryption algorithms. See “Quantum money from knots” (Shor et al.)
- Knots also have applications in biology. DNA reproduction requires enzymes called topoisomerases to unknot the two DNA strands formed by replication, transcription, and recombination; certain chemotherapy drugs work by halting this untangling process, which requires knowing what transformations untie certain knots.
- Knots are also applicable to synthetic chemistry. Bonding the same atoms in the shape of different knots creates completely different substances, often with unique properties.

What is a knot?

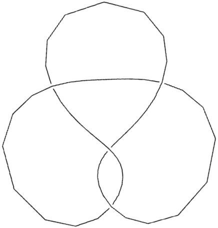
- Formally, a knot is an embedding of S^1 into $\mathbb{R}^3$. Two knots are equivalent if and only if there exists a continuous deformation from one knot to the other.
- Informally, we can think of knots as a rope: knot the rope, glue the ends of the rope together, and then make the rope infinitely thin.

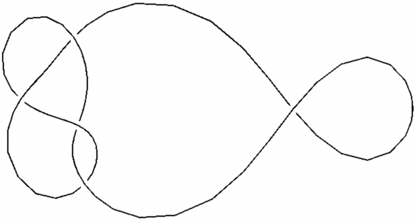
Pictured here is the figure-eight knot:



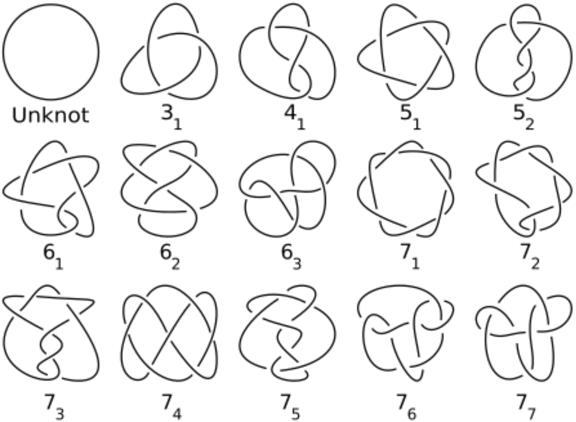
The Crossing Number

- Each of these 2D representations of a knot is a knot projection.
- Every point in the projection where multiple strands intersect is called a crossing.
- We can define the crossing number of knot K, denoted as $c(K)$, as the minimum # of crossings in a projection of the knot.
- If two knots having different crossing numbers, they must be the same knot. The two diagrams below are both projections of the same knot.





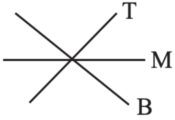
Knots Classified by $c(K)$

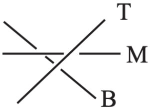


We can tabulate knots based on their crossing number. However, note that there are multiple knots K with $c(K) = x$ for all $x \geq 5$.

n-crossing diagrams and $c_n(K)$

$c(K)$ distinguishes some knots, but not all.
Are there other ways to distinguish them? Yes!
$c_n(K)$ for $n = 3, 4, 5, ...$ gives infinitely many.
Note: $c_2(K) = c(K)$.





- A standard crossing in a knot diagram is when one strand passes over another.
- An n-crossing projection is a knot diagram where at each crossing n strands intersect, with each strand bisecting the crossing.
- Every knot has an n-crossing projection for all $n \geq 2$. One can then define $c_n(K)$ as the smallest number of crossings an n-crossing projection of K can contain.

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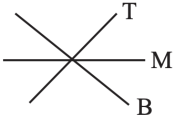

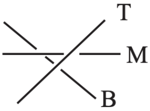

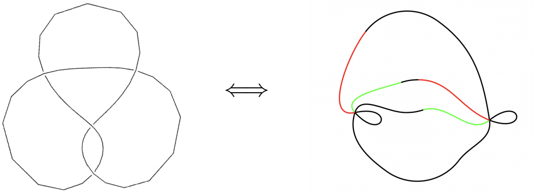

The figure shows a 3-crossing projection of the figure-eight knot with only two crossings. In fact, $c_3(4_1) = 2$. A green strand is the bottom strand and a red strand is the upper strand.

Inequalites for the n-crossing number

Computing $c_n(K)$ is very difficult. We need to find relationships between the n-crossing numbers to aid their computation. A few have been found:

- $c_{n+2}(K) \leq c_n(K)$
- $c_3(K) \leq c_2(K) - 1$ (for non-trivial knot K)
  Colin Adams (Williams College), 2013
- $c_4(K) \leq c_2(K) - 1$ (for non-trivial knot K)
  Michael Landry (Washington University), 2014
- $c_5(K) \leq c_3(K) - 1$ (for non-trivial knot K)
  Colin Adams (Williams College), Jim Hoste (Pitzer College), Martin Palmer (Universität Bonn), 2019
- My Theorem: $c_9(K) \leq c_3(K) - 2$ (for knot K that is not the trivial, trefoil, or figure-eight knot)
  (Hagedorn, 2023)

Please refer to my research paper for the proof of my theorem.

Implications

- The $c_9(K) \leq c_3(K) - 2$ inequality is the fifth known inequality between n-crossing numbers and is the first known inequality with an n-crossing number for $n > 5$.
- A major conjecture in the study of n-crossing diagrams states the following: $c_m(K) \leq c_n(K)$ for all $m \geq n$. My work provides further evidence in support of the conjecture.
- Using my results, I found the 9-crossing number for six knots whose 9-crossing number was previously unknown.
- My inequality cannot be improved as there exist knots K such that $c_3(K) = c_9(K) - 2$. For example, the $c_9(5_1) = c_3(5_1) - 2$ and $c_9(6_2) = c_3(6_2) - 2$.